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Conformal supergravity in twistor-string theory

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Abstract: Conformal supergravity arises in presently known formulations of twistor-string theory either via closed strings or via gauge-singlet open strings. We explore this sector of twistor-string theory, relating the relevant string modes to the particles and fields of conformal supergravity. We use the twistor-string theory to compute some tree level scattering amplitudes with supergravitons. Since the supergravitons interact with the same coupling constant as the Yang-Mills fields, conformal supergravity states will contribute to loop amplitudes of Yang-Mills gluons in these theories. Those loop amplitudes will therefore not coincide with the loop amplitudes of pure super Yang-Mills theory.

Keywords: Superstrings and Heterotic Strings, Topological Strings, Classical Theories of Gravity.

Contents

1. Introduction

The most commonly studied supergravity theories are "Einstein" supergravity theories, in which the gravitational part of the action, in n dimensions, is $\int d^n x\sqrt{g}R$, with R the Ricci scalar. These theories are not special to four dimensions; they exist up to eleven dimensions [1].

In four dimensions, there also exist conformally invariant supergravity theories (see [2] for a review), in which the gravitational part of the action is $\int d^4 x\sqrt{g}W^2$, with W the Weyl tensor. Such theories are special to four dimensions, in the following sense. In n dimensions, the conformally invariant expression would be $\int d^n x\sqrt{g}W^{n/2}$ if one does not consider terms involving derivatives of W.[1] Precisely for $n = 4$, this expression becomes quadratic in W and gives a nondegenerate kinetic energy for the gravitational field.

[1] In six dimensions, as pointed out to us by K. Skenderis, one can write a conformally invariant action of the form $\int d^6 x\sqrt{g}(W\Box W + \cdots)$, which leads to Euler-Lagrange equations that are of sixth order, as opposed to the fourth order equations coming from the usual conformally invariant action $\int d^n x\sqrt{g}W^{n/2}$.

This simple observation suggests that four-dimensional conformal supergravity theories might be relevant to the real world – perhaps with the aid of some mechanism of spontaneous breaking of conformal invariance. However, they in fact are generally considered to be an unsuitable starting point for describing nature, because they lead to fourth order differential equations for the fluctuations of the metric, and thus to a lack of unitarity. We have no reason to question these beliefs and we will later describe some facts that illustrate them.

The usual string theories are not conformally invariant in the target space, and give at low energies Einstein supergravity rather than conformal supergravity. Twistor-string theory [3, 4] is clearly different; superconformal invariance in spacetime is built in, and thus any form of supergravity that emerges will be conformal supergravity.

The first indication that conformal supergravity arises in twistor-string theory was presented in section 5.1 of [3], where it was shown that tree level gluon scattering amplitudes contain, in additional to the single-trace Yang-Mills scattering amplitudes, multi-trace terms that reflect the exchange of conformal supergravitons. At tree level, it is possible to recover the pure Yang-Mills scattering by extracting the single-trace amplitudes. However, at the loop level, diagrams that include conformal supergravitons can generate single-trace interactions, so the presence of conformal supergravity apparently means that it will be difficult with presently known forms of twistor-string theory to compute Yang-Mills or super Yang-Mills amplitudes beyond tree level. In twistor-string theory, conformal supergravitons have the same coupling constant as gauge bosons, so it is not possible to remove the conformal supergravity contributions to scattering amplitudes by going to weak coupling.[2] Since Yang-Mills theory makes sense without conformal supergravity, it is plausible that a version of twistor-string theory might exist that does not generate conformal supergravity and would be useful for computing Yang-Mills loop amplitudes. It would be highly desireable to find such a theory, though at the moment it is not clear how to do so.

The occurrence of conformal supergravity in twistor-string theory seems interesting and unusual enough to be worthy of study, even though conformal supergravity appears not to be physically sensible. In this paper, we consider two alternative versions of twistor-string theory based on the B-model of $\mathbb{CP}^{3|4}$ [3] and based on a certain construction involving open strings [4]. Since gravity usually arises in the closed string sector, the appearance of conformal supergravity in the open version of twistor-string theory is somewhat unexpected. (It would also be interesting to reconsider some of the issues using a more recent alternative twistor-string proposal [5], as well as the twistor-string proposal of [6].)

We consider various issues involving conformal supergravity in the two models, showing how rather similar results arise from different origins. In sections 2 and 3, we analyze the spectrum of the theory first in twistor space and then in Minkowski spacetime. In section 4, we discuss the linearized spectrum of conformal supergravity and compare to the twistor-string results. In section 5, we use the twistor-string theory to compute some tree

[2]To be more precise, in one approach to twistor-string theory [3] this is true as stated, while in the other approach [4], as we discuss in section 7, the ratio of the gravitational and gauge coupling constants is proportional to k, the level of the current algebra. Unitarity requires $k \geq 1$, but if it makes sense to relax unitarity, one could conceivably decouple conformal gravity in the limit $k \to 0$.

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amplitudes including conformal supergravitons. In section 6, we discuss some properties of the nonlinear conformal supergravity action. Finally, in section 7, we analyze anomalies that apparently lead to restrictions on the gauge group. Our discussion in section 7 is somewhat inconclusive.

2. Vertex operators

In this section, we will describe the twistor-string vertex operators. First we consider the "open string" version of twistor-string theory [4], and then we consider the B-model of $\mathbb{CP}^{3|4}$ [3].

2.1 "Open string" version

In the open version of the twistor string, the worldsheet action is

$$S = \int d^2 z (Y_I \overline{\partial}_A Z^I + \overline{Y}_I \partial_A \overline{Z}^I + S_C) \,. \tag{2.1}$$

Here, assuming a euclidean signature worldsheet,[3] Z^I are homogeneous coordinates of $\mathbb{CP}^{3|4}$, $\overline{Z}^I$ are their complex conjugates, and Y and $\overline{Y}$ are variables conjugate to Z and $\overline{Z}$, respectively. Z and $\overline{Z}$ have conformal dimensions $(0,0)$, while Y and $\overline{Y}$ have dimensions $(1,0)$ and $(0,1)$. A is a worldsheet gauge field that gauges the $GL(1,\mathbb{C})$ symmetry $Z^I \to tZ^I$, $Y_I \to t^{-1} Y_I$. S_C is the action for an additional system with $c = 28$. We assume that this includes a current algebra of some group G, which will become a gauge group in spacetime, and we refer to the variables in S_C as current algebra variables. For the open string,

$$Z^I = \overline{Z}^I \,, \qquad Y_I = \overline{Y}_I \tag{2.2}$$

on the boundary. On the boundary, the gauge group $GL(1,\mathbb{C})$ (or $GL(1,\mathbb{R})^2$ in the case of a Lorentz signature worldsheet, as remarked in the footnote) is broken to $GL(1,\mathbb{R})$, the group of real scalings of Z and $\overline{Z}$ that preserve the boundary condition.

Physical states are described by dimension one fields or vertex operators that are neutral under $GL(1)$ and moreover are primary fields with respect to the Virasoro and GL(1) generators. These generators are

$$T = Y_I \partial Z^I + T_C \,, \qquad J = Y_I Z^I \,, \tag{2.3}$$

where T_C is the $c = 28$ stress tensor for the current algebra. We consider mainly open string vertex operators. The boundary condition (2.2) means that open string vertex operators can be expressed in terms of Z and Y and the current algebra variables, and not $\overline{Z}$ and $\overline{Y}$; moreover, by virtue of the boundary condition, Z and Y are real on the boundary. (For a Lorentz signature worldsheet, Z and Y are real even away from the boundary, as noted in the footnote.)

[3]With a Lorentz signature worldsheet, one can alternatively treat Z^I as homogeneous coordinates of $\mathbb{RP}^{3|4}$, and $\overline{Z}^I$ as homogeneous coordinates of a second $\mathbb{RP}^{3|4}$. The gauge group is $GL(1,\mathbb{R}) \times GL(1,\mathbb{R})$, with separate real scalings of Z and $\overline{Z}$. The open string boundary condition is the same, $Z^I = \overline{Z}^I$, and the open string vertex operators are accordingly also the same.

The most obvious primary fields are the dimension zero fields $\phi(Z^I)$, with ϕ being any function that is invariant under $GL(1,\mathbb{R})$ scalings of Z^I (in other words, ϕ is invariant under $Z \to tZ$ for real t); equivalently, ϕ is any function on $\mathbb{RP}^{3|4}$. By multiplying such a field by any of the currents j_r, $r = 1, \ldots, \dim G$ of the current algebra, we can construct Yang-Mills vertex operators, which of course should have dimension 1:

$$V_\phi = j_r \phi^r(Z)\,. \tag{2.4}$$

These vertex operators were discussed in [4] in reproducing some of the results of [3] and [7].

With equal ease, one can construct the vertex operators that turn out to describe conformal supergravity. An expression linear in either Y or ∂Z has dimension 1. So the following operators have dimension 1:

$$V_f = Y_I f^I(Z)\,, \qquad V_g = g_I(Z)\partial Z^I\,. \tag{2.5}$$

These are in addition $GL(1)$-invariant if f^I carries GL(1) charge 1 (that is, under $Z \to tZ$, it scales as $f \to tf$) and g_I carries GL(1) charge -1 (it scales as $g \to t^{-1}g$). To be primary fields with respect to J and T, f^I and g_I must satisfy

$$\partial_I f^I = 0\,, \qquad Z^I g_I = 0. \tag{2.6}$$

Furthermore, f^I and g_I have the gauge invariances

$$\delta f^I = Z^I \Lambda, \quad \delta g_I = \partial_I \chi \tag{2.7}$$

since $Y_I Z^I \Lambda = J_{-1}\Lambda$ and $\partial Z^I \partial_I \chi = T_{-1}\chi$.

These conditions have a simple interpretation. Since f^I has charge 1, the expression

$$\Upsilon = f^I \frac{\partial}{\partial Z^I} \tag{2.8}$$

is invariant under scaling. The equivalence relation $\delta f^I = Z^I \Lambda$ means that Υ can be interpreted as a vector field on $\mathbb{RP}^{3|4}$ (and not on the ambient $\mathbb{R}^{4|4}$). The constraint $\partial_I f^I = 0$ means that it is a *volume-preserving* vector field.

In the case of g, the natural expression is the one-form

$$\Theta = g_I\, dZ^I\,. \tag{2.9}$$

The constraint $g_I Z^I = 0$ means that Θ is well-defined as a one-form on $\mathbb{RP}^{3|4}$ (and not just on $\mathbb{R}^{4|4}$). The gauge equivalence $\delta g_I = \partial_I \chi$ means that g can be regarded as an abelian gauge field on $\mathbb{RP}^{3|4}$, not just a one-form. Of course, like the functions ϕ^r in (2.4) that describe gauge fields in spacetime or the volume-preserving vector field f^I, the abelian gauge field g_J is not constrained to obey any equation of motion.

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2.2 The B-model Of $\mathbb{CP}^{3|4}$

Now we consider the analogous issues in the other version of twistor-string theory — the B-model of $\mathbb{CP}^{3|4}$.

We denote the complex homogeneous coordinates of $\mathbb{CP}^{3|4}$ as $Z^I = (\lambda^a, \mu^{\dot{a}}, \psi^A)$, where λ^a and $\mu^{\dot{a}}$ are bosonic spinors of opposite helicity, and ψ^A are fermions. $\mathbb{CP}'^{3|4}$ denotes the region in $\mathbb{CP}^{3|4}$ in which the λ^a are not both zero. The usual twistor space wavefunctions (for examples, wavefunctions corresponding to plane waves in Minkowski spacetime) are regular on $\mathbb{CP}'^{3|4}$.

The analysis in [3] centered on the open string states of the B-model that are related to gauge fields in spacetime. As suggested by the Penrose transform [8], they correspond to elements of the sheaf cohomology group $H^1(\mathbb{CP}'^{3|4}, \mathcal{O})$. Such an element is represented by a wavefunction which is a $(0,1)$-form $\tilde{\phi} = d\overline{Z}^I \omega_{\overline{I}}(Z, \overline{Z})$; it obeys $\overline{\partial}\tilde{\phi} = 0$, and is subject to the gauge equivalence $\tilde{\phi} \to \tilde{\phi} + \overline{\partial}\alpha$, for any function α on $\mathbb{CP}'^{3|4}$. $\tilde{\phi}$ and the gauge parameter α take values in the adjoint representation of the gauge group, though we have not shown this in the notation. Vertex operators for these states were described in [3].

The relation between the two ways of describing gauge fields in twistor space is described in section VI.5 of [9] and is as follows. Suppose that ϕ is a function on $\mathbb{RP}^{3|4}$. It is defined for real values of the ratio $z = \lambda^2/\lambda^1$. We assume that ϕ is real-analytic and so can be analytically continued to a neighborhood of $\mathbb{RP}^{3|4}$ in $\mathbb{CP}^{3|4}$; moreover, we assume that this neighborhood includes all points in $\mathbb{CP}'^{3|4}$ where z is real. Then we define $\tilde{\phi} = \phi \cdot \overline{\partial}\left(\vartheta(\mathrm{Im}\, z)\right)$ (where $\vartheta(\mathrm{Im}\, z)$ is equal to 1 for $\mathrm{Im}\, z > 0$ and 0 for $\mathrm{Im}\, z < 0$). The mapping $\phi \to \tilde{\phi}$ is the mapping from vertex operators that describe gauge fields in the open string approach to twistor-string theory to those that describe vertex operators in the B-model of $\mathbb{CP}^{3|4}$.

Now let us consider the conformal supergravity sector. In the B-model, the most obvious closed string mode is a deformation of the complex structure of $\mathbb{CP}'^{3|4}$. (The B-model, after all, is used to describe complex structure deformations in compactification of physical string theories on a Calabi-Yau threefold.) However, the deformation must preserve the holomorphic volume-form or measure, which we will call Ω, since Ω is part of the definition of the B-model. Let us describe what sort of deformations have this property. We cover $\mathbb{CP}'^{3|4}$ with open sets U_i which individually will be undeformed. (Explicitly, one can take two open sets, a set U_1 characterized by $\lambda^1 \neq 0$ and a set U_2 characterized by $\lambda^2 \neq 0$.) Then, one glues together the open sets U_i on their intersections U_{ij} via diffeomorphism of the form $Z^I \to Z^I + \epsilon f^I_{ij}$, where (to describe the deformation to first order) ϵ is an infinitesimal parameter. (Also, we define $f_{ji} = -f_{ij}$.) Here $f^I_{ij}\partial/\partial Z^I$ is a holomorphic vector field – so that the complex structures of U_i and U_j match together on their intersection. Moreover, so that the holomorphic measure Ω can be defined on the deformed manifold, f_{ij} must be a *volume-preserving* vector field. This means explicitly that

$$\frac{\partial}{\partial Z^I} f^I_{ij} = 0\,. \tag{2.10}$$

Finally, on triple intersections $U_i \cap U_j \cap U_k$, compatibility of the gluings requires that $f_{ij} + f_{jk} + f_{ki} = 0$. There are no such triple intersections in our explicit covering of $\mathbb{CP}'^{3|4}$

by two open sets, so in that example this condition is trivial. Taking all this together, the f's describe an element of the sheaf cohomology group $H^1(\mathbb{CP}'^{3|4}, T')$, where T' is the sheaf of volume-preserving vector fields.

In $\overline{\partial}$ cohomology, an element of this cohomology group is described by a wavefunction $\widehat{J} = d\overline{Z}^{\overline{I}} j_{\overline{I}}^K$ which obeys $\overline{\partial}\widehat{J} = 0$ (explicitly $\partial_{\overline{I}} j_{\overline{J}}^K - \partial_{\overline{J}} j_{\overline{I}}^K = 0$) and is volume-preserving, that is $\partial_K j_{\overline{I}}^K = 0$. $\widehat{J}$ is subject to the usual gauge equivalence $\widehat{J} \to \widehat{J} + \overline{\partial}\alpha$ for any section α of T'. The relation between $\widehat{J}$ and the corresponding object f^K in the open string case is easy to guess, by analogy with what we said in the gauge theory case: assuming f^K has a sufficient degree of analyticity, the relation is $d\overline{Z}^{\overline{I}} j_{\overline{I}}^K = f^K \cdot \overline{\partial}(\vartheta(\mathrm{Im}\, z))$.

Using the notation of eq. (4.11) of [3], a vertex operator corresponding to $\widehat{J}$ is

$$V_{\widehat{J}} = \eta^{\overline{I}} j_{\overline{I}}^K \theta_K \,. \tag{2.11}$$

(The operator we have written is a $(0,0)$-form; a $(1,1)$-form is obtained by the standard "descent" procedure.) Here $\eta^{\overline{I}}$ and θ_K are worldsheet fermions of the B-model. From this point of view, we do not understand why $\widehat{J}$ has to be volume-preserving.

The Penrose transform [10] shows that a volume-preserving deformation of the complex structure of twistor space describes a solution of the anti-self-dual Weyl equations in spacetime. This describes one helicity of conformal supergravity. Where does the other helicity come from?

It is plausible to postulate another type of closed string mode with vertex operator $b_{\overline{I}K}$ that couples to $D1$-branes via

$$\int_{\mathbb{D}} b_{\overline{I}K} d\overline{Z}^{\overline{I}} \wedge dZ^K \,, \tag{2.12}$$

where $\mathbb{D}$ is the world-volume of a $D1$-brane. In physical string theory, such a mode would arise in the RR sector (and might be called an RR B-field). On b, we can impose the standard equation of motion and gauge invariance of the B-model, $\overline{\partial}b = 0$, $b \to b + \overline{\partial}\lambda$. If these were the only conditions, b would be understood as an element of $H^1(\mathbb{CP}'^{3|4}, T^*)$, where T^* denotes the cotangent bundle. The coupling (2.12) is invariant under the additional transformation $b_{\overline{I}K} \to b_{\overline{I}K} + \partial_K w_{\overline{I}}$. We assume that b is subject to this additional invariance. b can then be related to the abelian gauge field g of the open string case by the familiar formula $d\overline{Z}^{\overline{I}} b_{\overline{I}K} = g_K \cdot \overline{\partial}(\vartheta(\mathrm{Im}\, z))$.

The vertex operator (2.11) is not automatically on-shell. We would like to postulate an effective action whose associated Euler-Lagrange equation places j on-shell. At the linearized level, the appropriate action is

$$\int_{\mathbb{CP}^{3|4}} d\overline{X}^{\overline{I}} d\overline{X}^{\overline{J}} d\overline{X}^{\overline{K}} b_{\overline{I}I} \partial_{\overline{J}} j_{\overline{K}}^I \Omega \,. \tag{2.13}$$

Here X^I are local complex coordinates on $\mathbb{CP}^{3|4}$ (as opposed to the homogeneous coordinates Z^I used in most of our formulas), and the complex conjugates $\overline{X}^I$ are purely bosonic. (Some of the X^I are fermionic, but as explained in [3], there is no need to introduce complex conjugates of the fermionic coordinates.) Upon varying with respect to b, (2.13) leads

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to $\overline{\partial} j = 0$ as an equation of motion. It is conceivable that one should introduce additional fields so that the condition for j to be volume-preserving would also arise as an equation of motion; however, we do not know a convenient way to do this. We also do not know how to explicitly show in the string theory the origin of the term (2.13) in the effective action.

While (2.13) is adequate to linear order in j, we would like to write a suitable action for complex structure deformations that is not limited to linear order. This can be done as follows. The "field" in the action will be an almost complex structure, which is a tensor $J^A{}_B$ constrained to obey $J^2 = -1$. The indices A and B can be either holomorphic or antiholomorphic. The unperturbed J is $J^I{}_J = i$, $J^{\overline{I}}{}_{\overline{J}} = -i$, with other components vanishing. The first order perturbation of J, subject to the constraints, has matrix elements $J^I{}_{\overline{J}}$ and $J^{\overline{I}}{}_J$; the components $J^I{}_{\overline{J}}$ are what we have hitherto called $j^I_{\overline{J}}$. From J one can construct an invariant tensor $N_{\overline{I}\,\overline{J}}{}^K$ called the Nijenhuis tensor; the almost complex structure J is called integrable if and only if $N = 0$. The linearized approximation to N is $N_{\overline{I}\,\overline{J}}{}^K = \partial_{\overline{I}} j_{\overline{J}}{}^K - \partial_{\overline{J}} j_{\overline{I}}{}^K$. The nonlinear extension of the action (2.13) is

$$\int_{\mathbb{CP}^{3|4}} d\overline{X}^{\overline{I}}\, d\overline{X}^{\overline{J}} d\overline{X}^{\overline{K}} b_{\overline{I}I} N^I_{\overline{J}\,\overline{K}} \Omega \,. \tag{2.14}$$

The equation of motion for b asserts that $N = 0$, or in other words the almost complex structure is integrable.

According to the Penrose transform [10], the condition $N = 0$ corresponds in Minkowski spacetime to $W_{abcd} = 0$, where W_{abcd} and $W_{\dot a\dot b\dot c\dot d}$, which are symmetric in all their indices, are the self-dual and anti-self-dual parts of the Weyl tensor. (To be more precise, in the case considered by Penrose, the result is $W_{abcd} = 0$, while in our situation, one will get a supersymmetric extension of this.) This strongly suggests that the spacetime interpretation of (2.14) would be an action $\int d^4x\sqrt{g}\, U^{abcd} W_{abcd}$ (or rather a supersymmetric extension of this [11]), where U^{abcd} (symmetric in all its indices) is a field of Lorentz spin $(2,0)$, just like W^{abcd}. As we discuss momentarily, U is part of the spacetime interpretation of the twistor field b.

To get conformal supergravity, we must, as in section 4 of [3], generate from D-instantons (which would correspond to worldsheet instantons in the other approach to twistor-string theory) a U^2 interaction. The resulting action

$$\int d^4x\sqrt{g}\left(U^{abcd}W_{abcd} - \frac{1}{2}\epsilon U^2\right) \tag{2.15}$$

is equivalent, after integrating out U, to $\frac{1}{2\epsilon}\int d^4x\sqrt{g} W^{abcd}W_{abcd}$, which is the action of conformal gravity.

To be more exact, the standard conformal gravity action is

$$\frac{1}{4\epsilon}\int d^4x\sqrt{g}\left(W^{abcd}W_{abcd} + W^{\dot a\dot b\dot c\dot d}W_{\dot a\dot b\dot c\dot d}\right).$$

As in the gauge theory case treated in [3], the two differ by

$$\frac{1}{4\epsilon}\int d^4x\sqrt{g}\left(W^{abcd}W_{abcd} - W^{\dot a\dot b\dot c\dot d}W_{\dot a\dot b\dot c\dot d}\right),$$

which is a topological invariant that does not affect perturbation theory.

In the present situation, it is not hard to see where the U^2 term will come from. We have already postulated the coupling (2.12) of the field b to $D1$-branes. This coupling implies that the contribution of a $D1$-brane is proportional to $\exp(-\int_{\mathbb{D}} b)$. Expanding this in powers of b and integrating over moduli, the part of the effective action quadratic in b is

$$\frac{1}{2}\int_{\mathcal{M}} d\mu \left(\int_{\mathbb{D}} b\right)^2 , \tag{2.16}$$

from which we will extract the U^2 term. $\mathcal{M}$ denotes the component of the moduli space of curves in $\mathbb{CP}^{3|4}$ that contains $\mathbb{D}$; the corresponding contribution to the effective action is evaluated by integrating over this moduli space with a suitable measure $d\mu$, as noted in (2.16).

The relevant case is the case that $\mathbb{D}$ is a degree one instanton, that is a copy of $\mathbb{CP}^1$ linearly embedded in $\mathbb{CP}^{3|4}$. In the Penrose transform, as reviewed in [3], such a $\mathbb{D}$ corresponds to a point in Minkowski spacetime, or more exactly, a point in a chiral Minkowski superspace with coordinates $x^{a\dot{a}}$, θ^{Aa}. Thus, we can define a function $\mathcal{W}(x,\theta)$ on Minkowski superspace by $\mathcal{W}(x,\theta) = \int_{\mathbb{D}_{x,\theta}} b$, where $\mathbb{D}_{x,\theta}$ is the curve with moduli x and θ. ($\mathcal{W}$ is holomorphic since $\overline{\partial} b = 0$.) The θ expansion of $\mathcal{W}$ is discussed in more detail in section 4,[4] but for now we simply note that it reads in part

$$\mathcal{W} = C + \cdots + \frac{1}{4!}\epsilon_{ABCD}\theta^{Aa}\theta^{Bb}\theta^{Cc}\theta^{Dd}U_{abcd} + \cdots . \tag{2.17}$$

Here U is the field we want, and C is a scalar that will turn out to be a chiral "dilaton."

The interaction (2.16) becomes

$$\int d^4x^{a\dot{a}} d^8\theta^{Aa}\, \mathcal{W}^2 . \tag{2.18}$$

Upon performing the theta integral using (2.17), we do get the desired $\int d^4x\sqrt{g}U^2$ coupling.

We can also now see that C behaves as a dilaton. Let k be a Kahler form of $\mathbb{CP}^{3|4}$, normalized so that for a degree one curve $\mathbb{D}$, $\int_{\mathbb{D}} k = 1$. Consider shifting b by $b \to b + ck$, where c is a complex constant. This shifts the scalar field C by a constant, $C \to C+c$. Now, let $\mathbb{D}'$ be any D-instanton of degree d. Then (by the definition of the degree) $\int_{\mathbb{D}'} k = d$, so under $C \to C+c$, $\int_{\mathbb{D}'} k \to \int_{\mathbb{D}'} k + dc$. It follows that under this shift, the contribution of $\mathbb{D}'$ to a scattering amplitude, which is proportional to $\exp(-\int_{\mathbb{D}'} b)$, is multiplied by $\exp(-dc)$. Differently put, if $\langle C\rangle$ denotes the expectation value of C, then the $\langle C\rangle$ dependence of a degree d contribution to the scattering amplitudes is

$$\exp(-d\langle C\rangle) . \tag{2.19}$$

This is not the complete story. Parity invariance implies that there must be another scalar field $\overline{C}$ with parity conjugate couplings; for example, $\overline{C}$ couples to $W^2_{\dot{a}\dot{b}\dot{c}\dot{d}}$ while C couples to W^2_{abcd}. $\overline{C}$ is the "top" component of the superfield $\mathcal{W}$, as we describe more fully in section 4.

[4] In section 4, the Weyl tensor appears instead of U, as the expansion is made in a framework in which U has been integrated out.

The results that we have just described have counterparts in the open string approach to twistor-string theory. In that context, the form of the vertex operator $V_g = g_I \partial Z^I$ shows that there is a coupling $\int_{\partial D} g_I dZ^I$ of the g-field to the boundary ∂D of an open string worldsheet D. If this boundary is a "line" $\mathbb{D}$ in $\mathbb{RP}^{3|4}$, corresponding to a point in real Minkowski superspace (of $++--$ signature), then the definition of the corresponding superspace field is $\mathcal{W} = \int_{\mathbb{D}} g_I dZ^I$.

3. Minkowski space interpretation

In this section, we determine the spectrum of massless fields in Minkowski spacetime that is associated with the twistor space vertex operators found in section 2. It does not matter which type of twistor-string theory we use, since the two types of wavefunction are related by a map ($\phi \to \phi \cdot \overline{\partial}(\vartheta(\mathrm{Im}\, z))$) that was described in section 2. For brevity, we will use the open string language in this section.

The basic input we need is that [8, 9] a function of the homogeneous coordinates Z^I of twistor space that is homogeneous in the Z^I of degree k describes a massless state in Minkowski spacetime of helicity $1 + k/2$.

As an example, let us consider the field $f^I(Z)$ found in section 2. This field is a function of bosonic and fermionic variables $\lambda^a, \mu^{\dot{a}}$, and ψ^A; let us first determine the fields we get if we set $\psi^A = 0$.

For each value of I, $f^I(\lambda, \mu)$ is homogeneous in Z of degree 1. So if we ignore the spin carried by the I index, we get four bosonic and four fermionic helicity states, each of helicity 3/2.

Of course, it is not correct to ignore the spin carried by the I index. The possible choices of I are $(\alpha, \dot{\alpha}, A)$, where the cases $I = \alpha$ or $\dot{\alpha}$ are bosonic states, and $I = A$ are fermionic states. Both α and $\dot{\alpha}$ take two possible values and, under rotation around any given direction in space (the relevant direction is the direction of motion of a massless state in Minkowski spacetime corresponding to $f^I(Z)$), these two states have helicity 1/2 and $-1/2$. So from $f^I(\lambda, \mu)$, before taking account of the constraint and gauge-invariance, we get two bosonic states of helicity 2 and two of helicity 1. As for the fermions, since the index A carries no helicity but transforms as **4** of the SU(4) group of R-symmetries, we get four states of helicity 3/2 and transforming in that representation.

The gauge invariance $f^I \to f^I + Z^I \Lambda$ tells us to discard the state described by a function $\Lambda(\lambda, \mu)$ that is homogeneous of degree zero — in other words, a bosonic state of helicity 1. The constraint $\partial_I f^I = 0$ likewise removes the state described by the function $\partial_I f^I$, which is homogeneous of degree zero and so describes another bosonic state of helicity 1. After removing these two, we are left with two bosonic states of helicity 2 and four helicity 3/2 fermions transforming in the **4** of $\mathrm{SU}(4)_R$.

Here is another way to do this counting for the bosonic states. The two functions f^a have the same content as the two Lorentz-invariant functions $\lambda_a f^a$ and $\partial f^a / \partial \lambda^a$, and likewise the $f^{\dot{a}}$ are equivalent to two more Lorentz-invariant functions $\mu_{\dot{a}} f^{\dot{a}}$ and $\partial f^{\dot{a}} / \partial \mu^{\dot{a}}$. These two functions of degree 2 and two of degree 0 again describe two bosonic states of helicity 2 and two of helicity 1. The two of helicity 1 are removed, as before, using the gauge invariance and constraint.

This gives us the states at $\psi = 0$. To get the full spectrum, we expand in powers of ψ: $f^I(\lambda,\mu,\psi) = f_0^I(\lambda,\mu) + f_{1\,A}^I(\lambda,\mu)\psi^A + f_{2\,AB}^I(\lambda,\mu)\psi^A\psi^B + \cdots$. Here f_k^I is homogeneous in λ,μ with degree $1-k$, and so describes a massless state of helicity $3/2-k/2$ if we ignore the angular momentum carried by the I index. Upon taking that angular momentum into account, as well as the gauge-invariance and the constraints, we get the full collection of helicity states described by the field $f^I(Z)$:

$$\begin{aligned} \lambda^a f_a:&\quad (2,\mathbf{1}),(3/2,\overline{\mathbf{4}}),(1,\mathbf{6}),(1/2,\mathbf{4}),(0,\mathbf{1}) \\ \mu^{\dot a} f_{\dot a}:&\quad (2,\mathbf{1}),(3/2,\overline{\mathbf{4}}),(1,\mathbf{6}),(1/2,\mathbf{4}),(0,\mathbf{1}) \\ f^A:&\quad (3/2,\mathbf{4}),(1,\mathbf{15}\oplus\mathbf{1}),(1/2,\overline{\mathbf{20}}\oplus\overline{\mathbf{4}}),(0,\mathbf{10}\oplus\mathbf{6}),(-1/2,\mathbf{4})\,. \end{aligned} \tag{3.1}$$

Here the first entry is the helicity, and the second is the $\mathrm{SU}(4)_R$ representation.

Next, we perform a similar analysis for $g_I(Z)$, first considering the fields that arise at $\psi^A = 0$. The Lorentz scalars

$$(Z^a g_a,\ Z^{\dot a} g_{\dot a},\ \partial_a g^a,\ \partial_{\dot a} g^{\dot a}) \tag{3.2}$$

are homogeneous of degrees

$$(0,0,-2,-2)\,. \tag{3.3}$$

To allow for the gauge-invariance $g_I \to g_I + \partial_I\Lambda$ and the constraint $Z^I g_I = 0$, we should remove the two fields of degree 0. So we are left with two twistor space fields of degree -2, describing states in Minkowski space of helicity 0. And the fields g_A are homogeneous of weight -1 and therefore describe massless fields with helicity $+1/2$.

Allowing for the dependence on ψ^A, the complete massless superfields described by g_I are

$$\begin{aligned} \partial_a g^a:&\quad (0,\mathbf{1}),(-1/2,\overline{\mathbf{4}}),(-1,\mathbf{6}),(-3/2,\mathbf{4}),(-2,\mathbf{1}) \\ \partial_{\dot a} g^{\dot a}:&\quad (0,\mathbf{1}),(-1/2,\overline{\mathbf{4}}),(-1,\mathbf{6}),(-3/2,\mathbf{4})\,,(-2,\mathbf{1}) \\ g_A:&\quad (1/2,\overline{\mathbf{4}}),(0,\overline{\mathbf{10}}\oplus\mathbf{6}),(-1/2,\mathbf{20}\oplus\mathbf{4}),(-1,\mathbf{15}\oplus\mathbf{1}),(-3/2,\overline{\mathbf{4}})\,. \end{aligned} \tag{3.4}$$

The above results show that the massless fields described by g_I have the opposite helicities and conjugate SU(4) representations from those described by f^I, permitting these fields to be combined together in writing an action, as proposed in section 2.2. This duality between f and g can be understood by defining the Fourier-like transform[5]

$$\widetilde{g}^I(V) = \int_{\mathbb{RP}^{3|4}} d\Omega \exp(Z^K V_K) f^I(Z)\,. \tag{3.5}$$

Here the V_K are homogeneous coordinates on a new $\mathbb{RP}^{3|4}$ that is dual to the original one. The integral over $\mathbb{RP}^{3|4}$ is defined using the scaling-invariant measure $d\Omega$ (roughly

[5]The argument for parity symmetry in [12] involved a stringy extension of this Fourier transform.

$\lambda^a d\lambda_a\, d^2\mu\, d^4\psi$) described in [3].[6] Given that f^I is homogeneous of degree 1 in Z^I, $\widetilde{g}^I$ is homogeneous of degree -1 in V_I. Moreover, the Fourier transform maps the gauge-invariance $\delta f^I = Z^I\Lambda$ to a gauge-invariance $\delta\widetilde{g}^I = \partial^I\Lambda$, and the constraint $\partial_I f^I = 0$ to a constraint $V_I\widetilde{g}^I = 0$. In other words, the Fourier transform maps the f^I field in the original $\mathbb{RP}^{3|4}$ to a dual field $\widetilde{g}^I$ on the dual $\mathbb{RP}^{3|4}$. Since the Fourier transform is a $P\,\mathrm{SU}(4|4)$-invariant operation, it is clear that f^I describes spacetime fields with the same quantum numbers as those described by $\widetilde{g}^I$. Because the V_I have quantum numbers dual to those of the Z^I, $\widetilde{g}^I$ describes states with quantum numbers dual to those described by g_I, so g_I and f^I describe states with dual quantum numbers, as we have seen more laboriously above.

Lack of unitarity. So far we have exploited Lorentz-invariance in constructing the scalar functions $\sigma = \lambda_a f^a$ and $\sigma' = \mu_{\dot a} f^{\dot a}$. But we have not considered the rest of the Poincaré group. How, in fact, do the states transform under spacetime translations?

A twistor space function of definite homogeneity describes a particle state in Minkowski spacetime of definite helicity. Spacetime translations act on the Minkowski coordinates $x^{a\dot a}$ in the familiar fashion $x^{a\dot a} \to x^{a\dot a} + c^{a\dot a}$. In terms of the Z^I, the transformation is

$$\mu^{\dot a} \to \mu^{\dot a} + c^{a\dot a}\lambda_a\,. \tag{3.6}$$

The generator of this transformation is

$$D = c^{a\dot a}\lambda_a \frac{\partial}{\partial\mu^{\dot a}}\,. \tag{3.7}$$

To determine how a vector field $f^I\partial_I$ commutes with translations, we must evaluate the commutator $[D, f^I\partial_I]$. When we do this, we get two kinds of term. One term arises from $[D, f^I]$. This term describes the action of translations on f^I, as if the components of f^I were scalar functions on twistor space of the appropriate homogeneity. Let us write P for the operator that maps $f^I\partial_I$ to $[D, f^I]\partial_I$. There is also a second term which arises because ∂_I does not commute with D. This second term arises precisely if $I = a$, since $\partial/\partial\lambda^a$ does not commute with D (but $\partial/\partial\mu^{\dot a}$ and $\partial/\partial\psi^A$ do so commute).

The net effect is that on the pair $\left(\begin{smallmatrix} f^{\dot a} \\ f^a \end{smallmatrix}\right)$, the translation generator D acts as

$$\begin{pmatrix} P & * \\ 0 & P \end{pmatrix}, \tag{3.8}$$

where P would represent ordinary translations and the off-diagonal term $*$ arises from $[D, \partial_I] \neq 0$.

The matrix (3.8) is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are hermitean operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable.

[6] Our notation is slightly non-standard; we write Ω for the "volume-form" on twistor space and, when convenient, we write $d\Omega$ for the associated measure on real twistor space.

Clearly, on vector fields $f^{\dot a}\partial_{\dot a}$, corresponding to vertex operators $f^{\dot a}Y_{\dot a}$, the translation generators can be diagonalized. The same is true for vertex operators $f^A Y_A$. They correspond to plane waves in Minkowski spacetime. The spacetime interpretation of the other vertex operators $f^a Y_a$ will become clearer in the next section.

A similar analysis for the dual fields $g_I \partial Z^I$ shows that on the pair $\binom{g_{\dot a}}{g_a}$, the translations act as the transpose of (3.8). Hence the translations can be diagonalized in acting on vertex operators $g_a \partial Z^a$.

4. Spacetime interpretation

In this section, we first review the linearized description of conformal supergravity in superspace, [13, 14] then determine the corresponding spectrum of massless helicity states, and finally compare to the results of sections 2 and 3 based on twistor-string theory.

4.1 Linearized conformal supergravity in superspace

At the linearized level, $\mathcal{N}=4$ conformal supergravity can be described off-shell [13, 14] by a chiral scalar superfield $\mathcal{W}(x^{a\dot a}, \theta^A_a, \overline{\theta}^{\dot a}_A)$ which satisfies the condition

$$\epsilon^{ABCD} D^a_C D_{Ea} D_{bD} D^b_F \mathcal{W} = \epsilon_{EFGH} \overline{D}^{\dot a A} \overline{D}^G_{\dot a} \overline{D}^B_{\dot b} \overline{D}^{\dot b H} \overline{\mathcal{W}} . \tag{4.1}$$

Here D^a_A and $\overline{D}^A_{\dot a}$ are the usual superspace derivatives; it is convenient to choose coordinates with $D^a_A = \frac{\partial}{\partial \theta^A_a} + \overline{\theta}^{\dot a}_A \partial_{a\dot a}$, $\overline{D}^A_{\dot a} = \frac{\partial}{\partial \overline{\theta}^{\dot a}_A}$, so that the condition for $\mathcal{W}$ to be chiral, namely $\overline{D}^A_{\dot a}\mathcal{W} = 0$, reduces to the statement that $\mathcal{W}$ is independent of $\overline{\theta}^{\dot a}_A$. The field $\mathcal{W}$ has an analog in $\mathcal{N}=2$ super Yang-Mills theory; that theory is described [15] at the linearized level by an adjoint-valued chiral superfield $\mathcal{W}_{YM}$ satisfying

$$D^a_C D_{Da} \mathcal{W}_{YM} = \overline{D}^{\dot a}_C \overline{D}_{D\dot a} \overline{\mathcal{W}}_{YM} . \tag{4.2}$$

These conditions are customarily called reality conditions because they do in fact imply that certain components in the θ expansions of $\mathcal{W}$ and $\mathcal{W}_{YM}$ are real, while others obey Bianchi identities. For example, (4.2) imples that the auxiliary fields $d_{(CD)} = D^a_C D_{Da}\mathcal{W}_{YM}$ are real and that the Yang-Mills field strength, whose self-dual part is $F_{ab} = \epsilon^{AB} D_{Aa} D_{Bb} \mathcal{W}_{YM}$, satisfies the usual Bianchi identities. Similarly, (4.1) implies that the auxiliary fields $d^{[AB]}_{[EF]} = \epsilon^{ABCD} D^a_C D_{Ea} D_{bD} D^b_F \mathcal{W}$ are real and that the Weyl tensor, whose self-dual part is $W_{abcd} = \epsilon^{ABCD} D_{Aa} D_{Bb} D_{Cc} D_{Dd} \mathcal{W}$, satisfies the usual Bianchi identities.

The component field expansion of $\mathcal{W}$ is

$$\begin{aligned} \mathcal{W}(x,\theta) = {} & C + \theta^{Aa}\Lambda_{Aa} + (\theta^2)^{(AB)} E_{(AB)} + (\theta^2)^{(ab)}_{[AB]} T^{[AB]}_{(ab)} + (\theta^3)^{(abc)}_D (\partial\eta)^D_{(abc)} + \\ & + (\theta^3)^{aC}_{[AB]} \xi^{[AB]}_{aC} + (\theta^4)^{A(ab)}_B (\partial V)^B_{(ab)A} + (\theta^4)^{(abcd)} W_{abcd} + (\theta^4)^{[AB]}_{[CD]} d^{[CD]}_{[AB]} + \\ & + (\theta^5)^{a[AB]}_C \partial_{a\dot a} \overline{\xi}^{\dot a C}_{[AB]} + (\theta^5)^{A(abc)} (\partial\rho)_{A(abc)} + (\theta^6)_{(AB)} \partial_\mu \partial^\mu \overline{E}^{(AB)} + \\ & + (\theta^6)_{[AB](ab)} \partial^{a\dot a} \partial^{b\dot b} \overline{T}^{[AB]}_{(\dot a\dot b)} + (\theta^7)_{Aa} (\partial^\mu \partial_\mu) \partial^{a\dot a} \overline{\Lambda}^A_{\dot a} + (\theta^8)(\partial_\mu\partial^\mu)^2 \overline{C} . \end{aligned} \tag{4.3}$$

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By virtue of (4.1), the component fields in this expansion obey various conditions. W_{abcd}, which in the nonlinear theory is interpreted as the self-dual Weyl tensor, obeys Bianchi identities which imply that it can constructed from a vielbein $e_\mu^{a\dot a}$. d obeys $d_{[AB]}^{[BC]}=0$, as well as being real, and similarly $\xi^B_{[AB]}=0$. Bianchi identities following from (4.1) have been used to write certain components of the θ expansion of $\mathcal{W}$ in terms of potentials η, V, ρ; here, $V^A_{\mu A}=0$ and

$$\begin{aligned}(\partial V)^B_{(ab)A} &= \partial_\mu V^B_{\nu A}(\sigma^{\mu\nu})_{(ab)}\,,\\ (\partial\eta)^A_{(abc)} &= \partial_\mu \eta^A_{\nu(a}(\sigma^{\mu\nu})_{bc)}\,,\\ (\partial\rho)_{(abc)A} &= \partial_\mu \rho_{\nu A(a}(\sigma^{\mu\nu})_{bc)}\,.\end{aligned}\tag{4.4}$$

Moreover, $\rho_{\mu Aa}$ is related to $\overline{\eta}_{\mu A\dot a}$ by the formula

$$\rho_{\nu Aa}=\sigma^\mu_{a\dot a}\left(\partial_\mu\overline{\eta}^{\dot a}_{\nu A}-\partial_\nu\overline{\eta}^{\dot a}_{\mu A}+\frac{1}{2}\epsilon_{\mu\nu\tau\kappa}\partial^\tau\overline{\eta}^{\kappa\dot a}_A\right).\tag{4.5}$$

The linearized action for $\mathcal{N}=4$ conformal supergravity is

$$S=\int d^4x\int d^8\theta\;\mathcal{W}^2\,.\tag{4.6}$$

Upon performing the θ integrals, this gives the component action

$$\begin{aligned}S=&\int d^4x\Big(C(\partial^\mu\partial_\mu)^2\overline{C}+\Lambda_{aA}(\partial^\mu\partial_\mu)\partial^{a\dot a}\overline{\Lambda}^A_{\dot a}+E_{(AB)}\partial_\mu\partial^\mu\overline{E}^{(AB)}+\\ &+T^{[AB]}_{(ab)}\partial^{a\dot a}\partial^{b\dot b}\overline{T}_{(\dot a\dot b)[AB]}+\xi^{[AB]}_{aC}\partial^{a\dot a}\overline{\xi}^C_{\dot a[AB]}+d^{[AB]}_{[CD]}d^{[CD]}_{[AB]}+\\ &+(\partial V)^B_{(ab)A}(\partial V)^{(ab)A}_B+R_{(abcd)}R^{(abcd)}+(\partial\eta)^A_{(abc)}(\partial\rho)^{(abc)}_A\Big)\,.\end{aligned}\tag{4.7}$$

4.2 Spectrum of conformal supergravity

To find the helicities described by these fields, one needs to analyze solutions to their higher-derivative equations of motion. Though this is elementary and not essentially novel (see [16]), the details are slightly unfamiliar, because of the higher derivatives appearing in the kinetic operators.

For example, the equation of motion for the scalar field $C(x)$ is $\Box^2C=0$, where $\Box=\partial_\mu\partial^\mu$ is the usual kinetic operator for a massless scalar field. The equation $\Box C=0$ has for its general solution a superposition of plane waves $\exp(ik\cdot x)$, where $k^2=0$. Being of fourth order instead of second order, the equation $\Box^2C=0$ must have a general solution that depends on twice as many functions. Indeed, this equation is obeyed by the plane wave $\sigma_k=\exp(ik\cdot x)$, but also by $\sigma'_k=A\cdot x\exp(ik\cdot x)$, for any vector A. It may seem that we now have too many solutions, because of the choice of A. However, if $A\cdot k=0$, then σ'_k is a linear combination of the derivatives of σ_k with respect to k, and for purposes of constructing the general solution of $\Box^2C=0$, since we will include arbitrary linear combinations of the σ_k anyway, such choices of A are irrelevant. So for enumerating the possible solutions, we pick an A with $A\cdot k\neq 0$, and the precise choice of A does not matter.

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If now we consider how translations act on the pair of functions σ_k, σ'_k, we obtain a result that is in precise parallel with what we described from the twistor point of view at the end of section 3. If the translation operator $P_{a\dot{a}} = -i\partial/\partial x^{a\dot{a}}$ would act only on the plane waves $\exp(ik\cdot x)$ in the definition of σ_k and σ'_k, it would multiply those functions by $k_{a\dot{a}}$; because of the prefactor $A\cdot x$ in the definition of σ'_k, $P_{a\dot{a}}$ actually acts on the pair $\begin{pmatrix}\sigma_k\\ \sigma'_k\end{pmatrix}$ via the matrix

$$\begin{pmatrix} k_{a\dot{a}} & -iA_{a\dot{a}} \\ 0 & k_{a\dot{a}} \end{pmatrix}. \tag{4.8}$$

This matrix is undiagonalizable, which as we noted in section 3 reflects the nonunitarity of the theory.

Since the choice of A is arbitrary, we can take it to be a unit vector in the time direction. Making this choice, we write the general solution of the equation $\Box^2 C = 0$ as

$$\begin{aligned} C(x) = &\int d^3k\, e^{ik\cdot x + i|k|t}\left(C_0(|k|,k) + \frac{it}{|k|}C'_0(|k|,k)\right) + \\ &+ \int d^3k\, e^{ik\cdot x - i|k|t}\left(C_0(-|k|,k) - \frac{it}{|k|}C'_0(-|k|,k)\right). \end{aligned} \tag{4.9}$$

As in the case of an ordinary massless scalar field, this can be written more conveniently as

$$C(x) = \int d^4k\delta(k^2)e^{ik\cdot x}\left(C_0(k) + i\frac{x_0}{k_0}C'_0(k)\right), \tag{4.10}$$

where $C_0(k)$ and $C'_0(k)$, defined only for $k^2 = 0$, are independent fields of helicity zero. They make up what is called a "dipole." Note that the normalization of C'_0 has been chosen such that $\Box C = \int d^4k\delta(k^2)e^{ik\cdot x}C'_0(k)$.

The linearized action of conformal supergravity also contains a spinor field Λ^a_A with the third order equation of motion $\Box\partial_{a\dot{a}}\Lambda^a_A = 0$. We would like in a similar fashion to describe the helicity states of such a spinor field; we do this by describing the solutions of the equation whose wavefunctions are $\exp(ik\cdot x)$ times a polynomial. These are the states that have momentum k in the sense that they form a space (analogous to the space spanned by σ_k and σ'_k) in which the only eigenvalue of the translation operator is k.

To analyze these "plane waves," we introduce a pair of spinors $\pi, \tilde{\pi}$ such that $k^{a\dot{a}} = \pi^a\tilde{\pi}^{\dot{a}}$, and a second pair of spinors $\tau, \tilde{\tau}$ such that

$$\pi^a\tilde{\tau}_a = 1, \qquad \tilde{\pi}^{\dot{a}}\tau_{\dot{a}} = 1. \tag{4.11}$$

In particular, π and $\tilde{\tau}$ give a basis for the space of positive chirality spinors, so we can expand $\Lambda^a_A = \pi^a\Lambda^{(1)}_A + \tilde{\tau}^a\Lambda^{(2)}_A$. Since $\Box\partial_{a\dot{a}}\Lambda^a_A = 0$ implies that $\Box^2\Lambda^{(1)}_A = 0$ and that $\Box\Lambda^{(2)}_A = 0$, the most general solution is

$$\Lambda^a_A(x) = \int d^4k\delta(k^2)e^{ik\cdot x}\left(\pi^a(\Lambda_{-\frac{1}{2}A}(k) + i\frac{x_0}{k_0}\Lambda'_{-\frac{1}{2}A}(k)) + \tilde{\tau}^a\Lambda_{\frac{1}{2}A}(k)\right). \tag{4.12}$$

The plane wave $\pi^a\epsilon^{ik\cdot x}$ has helicity $-1/2$ (it is the standard plane wave solution of the ordinary chiral Dirac equation $\partial_{a\dot{a}}\Lambda^a = 0$!), and x_0/k_0 is invariant under spatial rotations.

So the terms $\Lambda_{-\frac{1}{2}A}$ and $\Lambda'_{-\frac{1}{2}A}$ describe waves of helicity $-1/2$. As $\tilde{\tau}$ transforms under rotations around the $\vec{k}$ axis oppositely to π, $\Lambda_{\frac{1}{2}A}$ describes a wave of helicity $+1/2$. $\Lambda_{-\frac{1}{2}A}$ and $\Lambda'_{-\frac{1}{2}A}$ combine to a "dipole" field of helicity $-1/2$, on which spacetime translations act in a nondiagonalizable fashion as in (4.8). On the other hand, $\Lambda_{\frac{1}{2}A}$ transforms as an ordinary field of helicity $+1/2$.

For $T^{[AB]}_{(ab)}$, the linearized equation of motion is $\partial^{a\dot{a}}\partial^{b\dot{b}}T^{[AB]}_{(ab)} = 0$. This equation similarly implies that

$$T^{[AB]}_{(ab)} = \int d^4k\delta(k^2)e^{ik\cdot x}\left(\pi_a\pi_b(T^{[AB]}_{-1}(k) + i\frac{x_0}{k_0}T'^{[AB]}_{-1}(k)) + \pi_{(a}\tilde{\tau}_{b)}T^{[AB]}_0(k),\right) \tag{4.13}$$

where on the right the subscript on $T^{[AB]}$ denotes the helicity.

The linearized equations obeyed by the gravitino $\eta^A_{\mu a}$ and the graviton $e_{\mu a\dot{a}}$ are

$$\partial^{e\dot{b}}\partial^{a(\dot{c}}\partial^{\dot{a})b}\sigma^{\mu}_{b\dot{b}}\eta^A_{\mu a} = 0, \quad \partial^{\dot{a}(c}\partial^{d)\dot{b}}\partial^{a(\dot{c}}\partial^{\dot{d})b}\sigma^{\mu}_{b\dot{b}}e_{\mu a\dot{a}} = 0\,. \tag{4.14}$$

By an analysis similar to the above, these equations imply that

$$\eta^A_{\mu a}{=}\sigma^{b\dot{b}}_{\mu}\int d^4k\delta(k^2)e^{ik\cdot x}\left(\pi_a\pi_b\tau_{\dot{b}}\left(\eta^A_{-\frac{3}{2}}(k){+}i\frac{x_0}{k_0}\eta^A_{-\frac{3}{2}'}(k)\right) + \pi_{(a}\tilde{\tau}_{b)}\tau_{\dot{b}}\eta^A_{-\frac{1}{2}}(k) + \tilde{\tau}_a\tilde{\tau}_b\tilde{\pi}_{\dot{b}}\eta^A_{\frac{3}{2}}(k)\right), \tag{4.15}$$

and that

$$\begin{aligned} e_{\mu a\dot{a}} = \sigma^{b\dot{b}}_{\mu}\int d^4k\delta(k^2)e^{ik\cdot x}\Big(&\pi_a\pi_b\tau_{\dot{a}}\tau_{\dot{b}}(e_{-2}(k) + i\frac{x_0}{k_0}e_{-2'}(k)) + \pi_{(a}\tilde{\tau}_{b)}\tau_{\dot{a}}\tau_{\dot{b}}e_{-1}(k) + \\ &+ \tilde{\tau}_a\tilde{\tau}_b\tilde{\pi}_{(\dot{a}}\tilde{\tau}_{\dot{b})}e_1(k) + \tilde{\tau}_a\tilde{\tau}_b\tilde{\pi}_{\dot{a}}\tilde{\pi}_{\dot{b}}(e_2(k) + i\frac{x_0}{k_0}e_{2'}(k))\Big)\,. \end{aligned} \tag{4.16}$$

To obtain this result, we have used the fact that polarizations of $\eta^A_{\mu a}$ and $e_{\mu a\dot{a}}$ which involve both π_a and $\tilde{\pi}_{\dot{a}}$ can be gauged away using the gauge transformations

$$\delta\eta^A_{\mu a} = \sigma^{b\dot{b}}_{\mu}(\pi_b\tilde{\pi}_{\dot{b}}\Omega^A_a + \epsilon_{ab}\kappa^A_{\dot{b}})\,, \qquad \delta e_{\mu a\dot{a}} = \sigma^{b\dot{b}}_{\mu}(\pi_b\tilde{\pi}_{\dot{b}}\Sigma_{a\dot{a}} + \epsilon_{ab}\tilde{l}_{\dot{a}\dot{b}} + \epsilon_{\dot{a}\dot{b}}l_{ab})\,,$$

where $(\Omega^A_a, \kappa^A_{\dot{b}}, \Sigma_{a\dot{a}}, \tilde{l}_{\dot{a}\dot{b}}, l_{ab})$ are gauge parameters.

Finally, the equations of motion for $E_{(AB)}$, $\xi^{[AB]}_{aC}$, and $V^B_{\mu A}$ are the usual equations for fields of helicity $0, 1/2$ and 1.

Combining these results, we summarize in table 1 the helicities and $\mathrm{SU}(4)_R$ representations of the physical states of $\mathcal{N} = 4$ conformal supergravity. In presenting table 1, we have also included a U(1) R-charge, defined so that $\mathcal{W}$ has charge 4 and θ has charge 1. The U(1) charges can be read off from the θ expansion of eq. (4.3). The analog of this U(1) charge in Yang-Mills theory is called S in [3]. The U(1) charge is not conserved by the interactions of conformal supergravity. In twistor-string theory, it is somewhat natural to add a constant to the U(1) generator, so that $\mathcal{W}$ has charge 0 while θ still has charge 1. Using this convention, which is adopted in [3], the linearized action (before integrating out auxiliary fields) has S charge -4, and D-instantons of degree d and genus g carry S charge $-4(1 + d - g)$. However, in table 1 we have simply defined the U(1) to be a symmetry of the linearized action.

	U(1) Charge	Helicity	SU(4) Representation
C	4	$0,0$	$\mathbf{1}$
Λ_A^a	3	$-\frac{1}{2},-\frac{1}{2},\frac{1}{2}$	$\overline{\mathbf{4}}$
$E_{(AB)}$	2	0	$\overline{\mathbf{10}}$
$T_{(ab)}^{[AB]}$	2	$-1,-1,0$	$\mathbf{6}$
$\xi_{aC}^{[AB]}$	1	$-\frac{1}{2}$	$\mathbf{20}$
η_μ^{Aa}	1	$-\frac{3}{2},-\frac{3}{2},-\frac{1}{2},\frac{3}{2}$	$\mathbf{4}$
$V_{\mu A}^B$	0	$1,-1$	$\mathbf{15}$
$d_{[AB]}^{[CD]}$	0	none	$\mathbf{20}$
$e_\mu^{a\dot{a}}$	0	$2,2,1,-1,-2,-2$	$\mathbf{1}$
$\overline{\eta}_{\mu A}^{\dot{a}}$	-1	$\frac{3}{2},\frac{3}{2},\frac{1}{2},-\frac{3}{2}$	$\overline{\mathbf{4}}$
$\overline{\xi}_{[AB]}^{\dot{a}C}$	-1	$\frac{1}{2}$	$\overline{\mathbf{20}}$
$\overline{T}_{(\dot{a}\dot{b})}^{[AB]}$	-2	$1,1,0$	$\mathbf{6}$
$\overline{E}^{(AB)}$	-2	0	$\mathbf{10}$
$\overline{\Lambda}_{\dot{a}}^A$	-3	$\frac{1}{2},\frac{1}{2},-\frac{1}{2}$	$\mathbf{4}$
$\overline{C}$	-4	$0,0$	$\mathbf{1}$

Table 1: U(1) charges, helicities, and $\mathrm{SU}(4)_R$ representations of physical states in $\mathcal{N}=4$ conformal supergravity in four dimensions. Whenever a field gives rise to two states with the same helicity, they form a "doublet," with the translation generators not being diagonalizable.

4.3 Identification with twistor fields

We can now verify that the gauge-singlet sector of twistor-string theory has the same physical states as conformal supergravity. Indeed, the results of table 1 for the physical states of $\mathcal{N}=4$ conformal supergravity in four dimensions agree with the results from eqs. (3.1) and (3.4) for the spacetime fields described by the twistor superfields f^I and g_I. Instead of simply making this comparison on a term by term basis, it is more illuminating to recognize that the chiral superfield $\mathcal{W}(x,\theta)$ that is the basic variable of linearized conformal supergravity coincides with the field of the same name

$$\mathcal{W}(x,\theta)=\int_{\mathbb{D}_{x,\theta}} g_I dZ^I \tag{4.17}$$

that we introduced at the end of section 3 in order to give a spacetime interpretation to the twistor fields. Here $\mathbb{D}_{x,\theta}$ is defined via the twistor equations

$$\left(\mu^{\dot{a}},\psi^A\right)=\left(x^{a\dot{a}}\lambda_a,\theta^{Aa}\lambda_a\right), \tag{4.18}$$

where now we take the variables to be real.

We can more explicitly write

$$\mathcal{W}(x,\theta)=\int d\lambda^a\left(g_a(Z)+x_{a\dot{a}}g^{\dot{a}}(Z)+\theta_a^A g_A(Z)\right). \tag{4.19}$$

Here we evaluated dZ^I using

$$\left(d\lambda^a,d\mu^{\dot{a}},d\psi^A\right)=\left(d\lambda^a,d\lambda_b x^{b\dot{a}},d\lambda_c\theta^{cA}\right), \tag{4.20}$$

which holds when the Z^I are evaluated on $\mathbb{D}_{x,\theta}$, or in other words are regarded as functions of λ^a for fixed x and θ via the twistor equations (4.18). On the right hand side of (4.19), these conditions on Z are understood.

The equation of motion for $\mathcal{W}$, derived from the superspace action (4.6), is

$$\epsilon^{ABCD} D^a_C D_{Ea} D_{bD} D^b_F \mathcal{W} = 0\,. \tag{4.21}$$

This implies but is stronger than the constraint (4.1). This equation of motion is automatically satisfied by any superspace function $\mathcal{W}$ that can be written in terms of a twistor field g_I via (4.17) or (4.19). To see this, first note that the equation of motion for $\mathcal{W}$ is equivalent to $\triangle_{AB}\triangle_{CD}\mathcal{W} = 0$, where $\triangle_{AB} = D_{Aa}D^a_B$. (In writing the equation for $\mathcal{W}$ this way, we appear to introduce extra equations absent in (4.21), but they are consequences of fermi statistics.) If $\phi(Z)$ is any function of Z, we can convert it to a function $\phi'(\lambda, x, \theta)$ by expressing μ and ψ in terms of λ, x, and θ via (4.18). (In other words, $\phi'(\lambda^a, x^{b\dot b}, \theta^{Ac}) = \phi(\lambda^a, x^{b\dot b}\lambda_b, \theta^{Ac}\lambda_c)$.) For any function ϕ' obtained this way, it follows from the chain rule that $D^b_F\phi' = \lambda^b(\partial_F + \overline{\theta}^{\dot b}_F\partial_{\dot b})\phi'$ where $\partial_F = \frac{\partial}{\partial\psi^F}$ and $\partial_{\dot b} = \frac{\partial}{\partial\mu^{\dot b}}$. From this it follows, using the fact that $\lambda_a\lambda^a = 0$, that $\triangle_{AB}\phi' = 0$. It does not follow from this that $\triangle_{AB}\mathcal{W} = 0$, because $\triangle_{AB}$ can act on the explicit factors of x and θ present on the right hand side of (4.19). However, since those factors are linear in x and θ, and $\triangle_{AB}$ annihilates $[\triangle_{CD}, x]$ and $[\triangle_{CD}, \theta]$, it does follow that $\triangle_{AB}\triangle_{CD}\mathcal{W} = 0$.

It is more difficult to write the twistor fields g_I in terms of $\mathcal{W}(x, \theta)$. As a step in this direction, first note that (4.17) implies that

$$\partial^\mu\partial_\mu\mathcal{W} = \int d\lambda^a(\partial^\mu\partial_\mu)x_{a\dot a}g^{\dot a}(\lambda, x\lambda, \theta\lambda) = \int d\lambda^a\lambda_a\partial_{\dot a}g^{\dot a}\,. \tag{4.22}$$

Using the description of the dipole fields in section 4 and the component expansion of $\mathcal{W}$ in (4.3), one therefore finds that

$$\partial_{\dot a}g^{\dot a}(Z) = \widehat{C}'_0 + \psi^A\widehat{\Lambda}'_{-\frac{1}{2}A} + (\psi^2)_{[AB]}\widehat{T}'_{-1}{}^{[AB]} + (\psi^3)_A\widehat{\eta}'^A_{-\frac{3}{2}} + \psi^4\widehat{e}'_{-2}\,, \tag{4.23}$$

where $\widehat{C}'_0$ denotes the twistor field of GL(1) charge -2 for the spacetime field C'_0 of helicity zero, $\widehat{\Lambda}'_{-\frac{1}{2}A}$ denotes the twistor field of GL(1) charge -3 for the spacetime field $\Lambda'_{-\frac{1}{2}A}$ of helicity $-\frac{1}{2}$, etc.

To obtain $g^{\dot a}$ from (4.23), one can use the usual momentum-space description of twistor fields in which $g^{\dot a}$ depends on $\mu^{\dot a}$ as $\delta(\pi^a\lambda_a)\exp(i\mu^{\dot a}\overline{\pi}_{\dot a}(\pi^1/\lambda^1))$, where $k^{a\dot a} = \pi^a\overline{\pi}^{\dot a}$ is the momentum and (π^1/λ^1) is Lorentz-covariant because of the delta-function $\delta(\pi^a\lambda_a)$ in $g^{\dot a}$. So $\partial_{\dot a}g^{\dot a} = i(\pi^1/\lambda^1)\overline{\pi}^{\dot a}g_{\dot a}$ and one can use the gauge invariance $\delta g_I = \partial_I\Omega$ to choose the gauge

$$g_{\dot a}(Z) = -i\frac{\lambda^a\sigma^0_{a\dot a}}{k^0}(\widehat{C}'_0 + \psi^A\widehat{\Lambda}'_{-\frac{1}{2}A} + (\psi^2)_{[AB]}\widehat{T}'_{-1}{}^{[AB]} + (\psi^3)_A\widehat{\eta}'^A_{-\frac{3}{2}} + \psi^4\widehat{e}'_{-2})\,. \tag{4.24}$$

Although this gauge choice is not Lorentz-covariant since it singles out the time direction, it is convenient for comparing with the dipole solution of (4.10) which also singles out the time direction.

To determine g_A, note that (4.17) implies that

$$\sigma^{\mu}_{a\dot{a}} \frac{\partial}{\partial \theta^A_a} \partial_\mu \mathcal{W} = \int d\lambda^a \lambda_a \left(\partial_A g_{\dot{a}}(\lambda, x\lambda, \theta\lambda) - \partial_{\dot{a}} g_A(\lambda, x\lambda, \theta\lambda) \right) . \tag{4.25}$$

Comparing with (4.3) and (4.12) and using that $\partial_{\dot{a}} g_A = i(\pi^1/\lambda^1)\overline{\pi}_{\dot{a}} g_A$, one finds that

$$\begin{aligned} g_A(Z) \, = \, & \widehat{\Lambda}_{\frac{1}{2}A} + \psi^B \widehat{T}_{0[AB]} + \psi^B \widehat{E}_{0(AB)} + (\psi^2)_{[AB]} \widehat{\eta}^B_{-\frac{1}{2}} + \\ & + (\psi^2)_{[BC]} \widehat{\xi}^{[BC]}_{-\frac{1}{2}A} + (\psi^3)_A \widehat{e}_{-1} + (\psi^3)_B \widehat{V}^B_{-1A} + \psi^4 \widehat{\overline{\eta}}_{-\frac{3}{2}A} \, . \end{aligned} \tag{4.26}$$

Note that this identification for $g_A(Z)$ is consistent with the equation

$$\epsilon_{ab} \frac{\partial}{\partial \theta^A_a} \frac{\partial}{\partial \theta^B_b} \mathcal{W} = \int d\lambda^a \lambda_a (\partial_A g_B(\lambda, x\lambda, \theta\lambda) + \partial_B g_A(\lambda, x\lambda, \theta\lambda)) \, , \tag{4.27}$$

which follows from (4.17).

Finally, one can use (4.17) to relate g_a with $\mathcal{W}$ by defining

$$\int d\lambda^a g_a = \mathcal{W} - \int d\lambda^a [x_{a\dot{a}} g^{\dot{a}} + \theta^A_a g_A] \, ,$$

which implies that

$$g_a(Z) = \lambda_a (\widehat{C}_0 + \psi^A \widehat{\Lambda}_{-\frac{1}{2}A} + (\psi^2)_{[AB]} \widehat{T}^{[AB]}_{-1} + (\psi^3)_A \widehat{\eta}^A_{-\frac{3}{2}} + \psi^4 \widehat{e}_{-2}) + \cdots \tag{4.28}$$

where $\ldots$ depends on fields appearing in $g_{\dot{a}}$ and g_A.

Similarly, one can relate the antichiral superfield $\overline{\mathcal{W}}(x + \theta\overline{\theta}, \overline{\theta})$ with the dual field $\widetilde{g}^I(\overline{Z})$ of (3.5) where $\overline{Z}_I$ plays the role of the dual variable V_I in (3.5). The identification is

$$\begin{aligned} \overline{\mathcal{W}}(\widehat{x}, \overline{\theta}) \, = \, & \int d\overline{Z}_I \widetilde{g}^I (\overline{Z}|_{\overline{\mu}^a = \widehat{x}^{a\dot{a}} \overline{\lambda}_a, \overline{\psi}_A = \overline{\theta}^{\dot{a}}_A \overline{\lambda}_{\dot{a}}}) \\ = \, & \int d\overline{\lambda}^{\dot{a}} \Big(\widetilde{g}_{\dot{a}}(\overline{\lambda}, \overline{\mu} = \widehat{x}\overline{\lambda}, \overline{\psi} = \overline{\theta\lambda}) + \widehat{x}_{a\dot{a}} \widetilde{g}^a(\overline{\lambda}, \overline{\mu} = \widehat{x}\overline{\lambda}, \overline{\psi} = \overline{\theta\lambda}) + \\ & \qquad + \overline{\theta}_{\dot{a}A} \widetilde{g}^A(\overline{\lambda}, \overline{\mu} = \widehat{x}\overline{\lambda}, \overline{\psi} = \overline{\theta\lambda}) \Big) \end{aligned} \tag{4.29}$$

where $\widehat{x}^{a\dot{a}} = x^{a\dot{a}} + \theta^{aA} \overline{\theta}^{\dot{a}}_A$. Using similar arguments to those above and defining f^I in terms of $\widetilde{g}^I$ using (3.5), (4.29) implies that one can choose a gauge such that

$$\begin{aligned} \lambda^a f_a(Z) \, = \, & \widehat{e}'_2 + \psi^A \widehat{\overline{\eta}}'_{\frac{3}{2}A} + (\psi^2)_{[AB]} \widehat{\overline{T}}'_1{}^{[AB]} + (\psi^3)_A \widehat{\overline{\Lambda}}'_{\frac{1}{2}}{}^A + \psi^4 \widehat{\overline{C}}'_0 \, , \\ f^A(Z) \, = \, & \widehat{\eta}^A_{\frac{3}{2}} + \psi^B \widehat{V}^B_{1A} + \psi^A \widehat{e}_1 + (\psi^2)^{[BC]} \widehat{\overline{\xi}}^A_{\frac{1}{2}[BC]} + \\ & + (\psi^2)^{[AB]} \widehat{\overline{\eta}}_{\frac{1}{2}B} + (\psi^3)_B \widehat{\overline{E}}_0^{(AB)} + (\psi^3)_B \widehat{\overline{T}}_0^{[AB]} + \psi^4 \widehat{\overline{\Lambda}}^A_{-\frac{1}{2}} \, , \\ f_{\dot{a}}(Z) \, = \, & \partial_{\dot{a}} (\widehat{e}_2 + \psi^A \widehat{\overline{\eta}}_{\frac{3}{2}A} + (\psi^2)_{[AB]} \widehat{\overline{T}}_1^{[AB]} + (\psi^3)_A \widehat{\overline{\Lambda}}^A_{\frac{1}{2}} + \psi^4 \widehat{\overline{C}}_0) + \cdots \, , \end{aligned} \tag{4.30}$$

where $\partial_a = \partial/\partial\lambda^a$, $\partial_{\dot{a}} = \partial/\partial\mu^{\dot{a}}$, and $\ldots$ depends on fields appearing in f_a and f^A.

5. Some tree-level scattering amplitudes

In this section, we evaluate some tree-level scattering amplitudes. First we consider three-point amplitudes and then MHV amplitudes. The computations are mainly done using the open-string version of twistor-string theory, but in section 5.3, we compare a few statements to analogous statements based on the B-model of $\mathbb{CP}^{3|4}$.

5.1 Three-point tree amplitudes

The three-point tree amplitudes are computed from the correlation function

$$\langle V_1(z_1)\ V_2(z_2)\ V_3(z_3)\rangle \tag{5.1}$$

at degree zero and degree one. We identify the open string worldsheet with the upper half of the complex z-plane; open string vertex operators are inserted on the real axis. We do not write the ghosts explicitly; as usual, they give a factor $|z_1 - z_2||z_2 - z_3||z_3 - z_1|$ that cancels a similar factor that arises in evaluating (5.1).

As explained in section 2.1, the vertex operators, for gauge bosons or for supergravitons of appropriate helicities, are respectively

$$V_\phi = j^r\phi_r(Z), \quad V_f = Y_I f^I(Z), \quad V_g = \partial Z^I g_I(Z)\,. \tag{5.2}$$

We first examine the degree zero contribution to the three-point functions. The correlators of the currents are familiar:

$$\langle j^r(z_1)j^s(z_2)j^t(z_3)\rangle = \frac{k f^{rst}}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}\,. \tag{5.3}$$

Here k is the level of the current algebra, and f^{rst} are the structure constants. What about the correlators of vertex operators constructed from fields on $\mathbb{RP}^{3|4}$? For functions $\phi_i(Z)$ on $\mathbb{RP}^{3|4}$, the degree zero correlator is simply

$$\left\langle \prod_{i=1}^{3}\phi_i(Z(z_i))\right\rangle = \int_{\mathbb{RP}^{3|4}} d\Omega \prod_i \phi_i(Z)\,, \tag{5.4}$$

where $d\Omega$ is the usual measure. Let us now try to evaluate a degree zero correlator containing a single Y field. The basic vertex operator containing the Y field is $f^I(Z)Y_I$, where as in section 2.1, $f^I(Z)$ is a volume-preserving vector field on $\mathbb{RP}^{3|4}$. Using $\langle Y_I(z)Z^J(w)\rangle = \delta_I^J/(z-w)$, we get

$$\left\langle f^I(Z)Y_I(z)\prod_{j=1}^{n}\phi_j(w_j)\right\rangle = \int_{\mathbb{RP}^{3|4}} d\Omega \sum_{j=1}^{n}\frac{1}{z-w_j} f^I(Z)\frac{\partial\phi_j(Z)}{\partial Z^I}\prod_{k\neq j}\phi_k(Z)\,. \tag{5.5}$$

Let us verify the $\mathrm{SL}(2,\mathbb{R})$ invariance of this formula. Under

$$z \to \frac{(az+b)}{(cz+d)}\,, \qquad w_i \to \frac{(aw_i+b)}{(cw_i+d)}\,, \tag{5.6}$$

(5.5) transforms to

$$\int_{\mathbb{RP}^{3|4}} d\Omega \sum_{j=1}^{n} (cz+d)(cw_j+d)\frac{1}{z-w_j} f^I(Z)\frac{\partial\phi_j(Z)}{\partial Z^I}\prod_{k\neq j}\phi_k(Z). \tag{5.7}$$

We would like the amplitude (5.5) under $\mathrm{SL}(2,\mathbb{R})$ to be simply multiplied by $(cz+d)^2$, that is, to transform to

$$(cz+d)^2\int_{\mathbb{RP}^{3|4}} d\Omega \sum_{j=1}^{n} \frac{1}{z-w_j} f^I(Z)\frac{\partial\phi_j(Z)}{\partial Z^I}\prod_{k\neq j}\phi_k(Z)\,, \tag{5.8}$$

reflecting the fact that the operator $f^I(Z)Y_I$ has conformal dimension 1 and the other operators have conformal dimension 0. The difference is

$$\begin{aligned}-c(cz+d)\int_{\mathbb{RP}^{3|4}} d\Omega \sum_{j=1}^{n} f^I(Z)\frac{\partial\phi_j(Z)}{\partial Z^I}\prod_{k\neq j}\phi_k(Z) &=\\ = -c(cz+d)\int_{\mathbb{RP}^{3|4}} d\Omega\, f^I(Z)\frac{\partial}{\partial Z^I}\prod_{k}\phi_k(Z).\end{aligned} \tag{5.9}$$

This vanishes upon integrating by parts and using the fact that f^I is volume-preserving, $\partial_I f^I = 0$.

Correlators with several Y fields are evaluated similarly by summing over contractions. For example,

$$\begin{aligned}\langle f_1^I Y_I(z_1) f_2^J Y_J(z_2) f_3^K Y_K(z_3)\rangle = &\frac{1}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}\times\\ &\times\int_{\mathbb{RP}^{3|4}}\left(\frac{\partial}{\partial Z^K}f_1^I\frac{\partial}{\partial Z^I}f_2^J\frac{\partial}{\partial Z^J}f_3^K -\right.\\ &\left.-\frac{\partial}{\partial Z^J}f_1^I\frac{\partial}{\partial Z^K}f_2^J\frac{\partial}{\partial Z^I}f_3^K\right).\end{aligned} \tag{5.10}$$

Degree zero correlators containing a field ∂Z vanish unless there is a Y field to contract it with and are evaluated using $\langle Y_I(z)\partial Z^J(w)\rangle = \delta_I^J/(z-w)^2$.

Using these rules, the nonvanishing degree zero three-point functions come from

$$\langle V_\phi V_\phi V_f\rangle\,,\qquad \langle V_f V_f V_f\rangle\,,\qquad \langle V_f V_f V_g\rangle\,. \tag{5.11}$$

For instance, we evaluated the $\langle V_f V_f V_f\rangle$ correlator in (5.10).

A perplexing point, which corresponds to facts noted in section 3.2 of [3],[7] is that, combining (5.3) and (5.4), the correlator $\langle V_{\phi_1}(z_1)V_{\phi_2}(z_2)V_\phi(z_3)\rangle$ is nonzero but antisymmetric in 1, 2, and 3, as a result of which, allowing for Bose statistics (and including the ghosts), the three-point coupling vanishes after summing over different cyclic orderings of z_1, z_2, and z_3. This three-point function would correspond to the three-gluon tree amplitude with helicities $++-$ (and other amplitudes related to this by supersymmetry). This amplitude

[7] See also a footnote on p. 7 of [12].

is non-vanishing in Yang-Mills theory for generic on-shell complex momenta, though it vanishes on-shell for real momenta in Lorentz signature. As explained in section 4.3 of [3], this amplitude is nonzero in the complex version of twistor space. (In that context, the fields carry an extra antiholomorphic index, which avoids the problem with Bose statistics.)

The above correlators lead to a variety of three-point amplitudes for helicity states. Among others, these include three-point amplitudes for states $A_{\pm 1}$ of the Yang-Mills gauge field with helicities ± 1, states $e_{\pm 2}$ of the graviton with helicities ± 2, and helicity zero states of the scalar fields C and $\overline{C}$. If we allow ourselves the liberty of including the $\langle V_\phi V_\phi V_\phi\rangle$ coupling whose anomalous status was noted in the last paragraph, then the correlators identified above lead to the following couplings:

$$A_1 A_1 A_{-1}\,, \qquad A_1 A_1 \overline{C}\,, \qquad A_1 A_{-1} e_2\,, \qquad \overline{C} e_2 e_2\,, \qquad e_2 e_2 e_{-2}\,, \qquad \overline{C} e_2 C\,. \tag{5.12}$$

To reach this conclusion, we use the results of section 3 (and of [3] in the case of V_ϕ). Vertex operators V_ϕ, V_f, and V_g describe supermultiplets whose bottom and top components are respectively $A_1 + \cdots + \psi^4 A_{-1}$, $e_2 + \cdots + \psi^4 \overline{C}$, and $C + \cdots + \psi^4 e_{-2}$. Evaluation of degree zero correlators entails an integral over $\mathbb{RP}^{3|4}$, as seen in all the formulas above; this picks out, among other things, amplitudes with two bottom components and a top component.

Now we move to degree 1. As explained in [3], section 3.1, a degree 1 curve is a "line" $\mathbb{D}_{x,\theta}$. It has homogeneous coordinates λ^a and is described by the familiar equations $\mu^{\dot a} = x^{a\dot a}\lambda_a$, $\psi^A = \theta^{Aa}\lambda_a$, where $x^{a\dot a}$ and θ^{Aa} are the moduli of the curve $\mathbb{D}_{x,\theta}$. The string worldsheet, which we describe as the complex z-plane, is mapped to $\mathbb{D}_{x,\theta}$ by $\lambda^1 = az + b$, $\lambda^2 = cz + d$, where a, b, c, and d are specified up to a constant multiple. In computing a degree 1 contribution to a scattering amplitude of open strings, we must as always impose a gauge condition to fix the $\mathrm{SL}(2,\mathbb{R})$ invariance of the open string worldsheet. While this can be done by imposing the condition that three of the open strings are inserted at specified values of z, say $0, 1$, and ∞, the instanton computation is simplest if one imposes $\mathrm{SL}(2,\mathbb{R})$ invariance by parametrizing the instanton as $\lambda^1 = 1$, $\lambda^2 = z$, and integrates over all insertion positions for the vertex operators. The degree 1 contribution to the scattering amplitude is thus computed from the formula

$$\int d^4x\, d^8\theta \int_{\mathbb{D}_{x,\theta}} dz_1 \dots dz_n \langle V_1(z_1)\dots V_n(z_n)\rangle\,. \tag{5.13}$$

Using this recipe, it is straightforward to identify the following nonzero three-point functions:

$$\langle V_\phi V_\phi V_\phi\rangle\,, \qquad \langle V_\phi V_\phi V_g\rangle\,, \qquad \langle V_g V_g V_g\rangle\,, \qquad \langle V_g V_g V_f\rangle\,. \tag{5.14}$$

Very few contractions of $\mathbb{RP}^{3|4}$ fields (as opposed to currents of the current algebra) are needed to evaluate these correlators; in fact, the only such contraction is a $\langle Y\partial Z\rangle$ contraction that is needed to evaluate $\langle V_g V_g V_f\rangle$. For example, $\langle V_g V_g V_g\rangle$ can be evaluated in degree 1 without any quantum contraction at all; it just equals $\int d^4x d^8\theta (\int_{\mathbb{D}_{x,\theta}} g_I dZ^I)^3 = \int d^4x\, d^8\theta\, \mathcal{W}^3$, a formula that was essentially explained in section 2.2.

Concentrating on the same helicities as before, these amplitudes describe the cubic couplings

$$A_{-1} A_{-1} A_1\,, \qquad A_{-1} A_{-1} C\,, \qquad C e_{-2} e_{-2}\,, \qquad e_{-2} e_{-2} e_2\,, \qquad C e_{-2} \overline{C}\,. \tag{5.15}$$

These results arise because the degree one curve $\mathbb{D}_{x,\theta}$ has eight fermionic moduli θ^{Aa}; integration over them generates amplitudes with two top components in the supermultiplets. The amplitudes in (5.15) are parity conjugates of the couplings that appeared at degree zero.

The cubic couplings we have obtained are consistent with the action

$$S=\int d^4x\left(\exp(2\overline{C})(F_{\dot a\dot b}F^{\dot a\dot b}+W_{\dot a\dot b\dot c\dot d}W^{\dot a\dot b\dot c\dot d}+\Box^2 C)+\exp(2C)(F_{ab}F^{ab}+W_{abcd}W^{abcd}+\Box^2\overline{C})\right). \tag{5.16}$$

This will be qualitatively compared in section 6 to expectations from conformal supergravity. (We have here included the kinetic energy of the C field in the form suggested by the discussion in section 6, though, as it leads to vanishing on-shell $CC\overline{C}$ and $C\overline{C}\,\overline{C}$ couplings, it is not detected by the above computation.)

5.2 MHV tree amplitudes

In this section, by evaluating (5.13), we will compute MHV tree amplitudes that include supergravitons in addition to possible gauge bosons. We will not compute all such amplitudes, but we will compute a set of them that contains amplitudes for gluons and for gravitons of either helicity as well as for the dilatonic fields C and $\overline{C}$.

We recall from section 2.1 that supergravitons are described by vertex operators $V_f = f^I(Z)Y_I$ and $V_g = g_I(Z)dZ^I$, where f^I is a volume-preserving vector field on $\mathbb{RP}^{3|4}$, and g_I is an abelian gauge field on $\mathbb{RP}^{3|4}$. To compute amplitudes with external gravitons and dilatons, it suffices, as we know from our analysis of the spectrum in section 3, to consider the components f^a, $f^{\dot a}$ of f^I (as opposed to f^A, which describes other modes), and similarly it suffices to consider the components g_a, $g_{\dot a}$ of g_I (as opposed to g_A).

In addition, to keep things simple, we will take external supergravitons to have wavefunctions that are plane waves $\exp(ik\cdot x)$, as opposed to the more general wavefunctions $A\cdot x\,\exp(ik\cdot x)$ encountered in section 4. Plane waves are wavefunctions on which the translations can be diagonalized. As we noted at the end of section 3, the vertex operators with this property are $f^{\dot a}Y_{\dot a}$ and $g_a\partial Z^a$.

Twistor space wavefunctions that correspond to plane waves in Minkowski spacetime have been described most fully (for gluons) in section 2.1 of [17]. Before reviewing these wavefunctions in the next paragraph, we recall some standard conventions. If α and β are two positive chirality spinors, we write $\langle\alpha,\beta\rangle$ as an abbreviation for $\epsilon_{ab}\alpha^a\beta^b$. Similarly, if $\tilde\alpha^{\dot a}$ and $\tilde\beta^{\dot b}$ have negative chirality, we write $[\tilde\alpha,\tilde\beta]$ for $\epsilon_{\dot a\dot b}\tilde\alpha^{\dot a}\tilde\beta^{\dot b}$. Given massless particles of momenta $p^i_{a\dot a}=\pi^i_a\tilde\pi^i_{\dot a}$, we also write $\langle i,j\rangle$ for $\langle\pi_i,\pi_j\rangle$, and $[i,j]$ for $[\tilde\pi_i,\tilde\pi_j]$.

The twistor space wavefunction of a massless Yang-Mills particle with definite momentum $p_{a\dot a}=\pi_a\tilde\pi_{\dot a}$ is $V_\phi=\sum_r\phi_r(\lambda,\mu,\psi)\cdot j^r$, where j^r are the currents, and roughly speaking each ϕ^r is a multiple of[8]

$$\phi(\lambda,\mu,\psi)=\delta(\langle\lambda,\pi\rangle)\exp(i[\mu,\tilde\pi])u(\psi)\,. \tag{5.17}$$

[8]See [3, 17] for more information on why this type of wavefunction describes a plane wave in spacetime. Roughly speaking, it describes a state of definite π because of the delta function, and a state of definite $\tilde\pi$ because of the exponential dependence on μ, so it describes a state of definite momentum $p_{a\dot a}=\pi_a\tilde\pi_{\dot a}$.

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Here the delta function has support on the locus where (up to scaling) $\lambda^a = \pi^a$, and the fermionic wavefunction $u(\psi)$ determines which helicity state we get in the Yang-Mills multiplet. Actually, (5.17) needs to be corrected slightly to get the right homogeneity in all variables. Since on the support of the delta function, λ is a multiple of π, there is a well-defined ratio (π/λ). The refined version of (5.17) that we really want is

$$\phi(\lambda,\mu,\psi) = \left(\frac{\lambda}{\pi}\right)\delta(\langle\lambda,\pi\rangle)\cdot\exp\left(i[\mu,\tilde{\pi}]\left(\frac{\pi}{\lambda}\right)\right)\cdot u\left(\left(\frac{\pi}{\lambda}\right)\psi\right). \tag{5.18}$$

The factors of (π/λ) make everything scale properly. The wavefunction is homogeneous in twistor coordinates $Z^I = (\lambda,\mu,\psi)$ of degree zero (this is the right scaling for ϕ, as we recall from section 2.1).

Plane wave states of supergravitons with momentum $p_{a\dot{a}} = \pi_a\tilde{\pi}_{\dot{a}}$ can be described by analogous twistor space wavefunctions. One type of vertex operator is $V_f = f^{\dot{a}}Y_{\dot{a}}$, where

$$f^{\dot{a}}(\lambda,\mu,\psi) = \tilde{\pi}^{\dot{a}}\left(\frac{\lambda}{\pi}\right)^2\delta(\langle\lambda,\pi\rangle)\cdot\exp\left(i[\mu,\tilde{\pi}]\left(\frac{\pi}{\lambda}\right)\right)\cdot u\left(\left(\frac{\pi}{\lambda}\right)\psi\right). \tag{5.19}$$

We take $f^{\dot{a}} \sim \tilde{\pi}^{\dot{a}}$ in order to satisfy the volume-preserving condition $\partial f^{\dot{a}}/\partial\mu^{\dot{a}} = 0$. The factors of (π/λ) ensure that $f^{\dot{a}}$ is homogeneous in twistor coordinates of weight one. The other vertex operator we need is $V_g = g_a\partial Z^a$, where

$$g_a(\lambda,\mu,\psi) = \lambda_a\left(\frac{\pi}{\lambda}\right)\delta(\langle\lambda,\pi\rangle)\cdot\exp\left(i[\mu,\tilde{\pi}]\left(\frac{\pi}{\lambda}\right)\right)\cdot u\left(\left(\frac{\pi}{\lambda}\right)\psi\right). \tag{5.20}$$

In this case, we take $g_a \sim \lambda_a$ to obey the constraint $\lambda^a g_a = 0$. The factors of (π/λ) ensures that g_a is homogeneous in twistor variables of degree -1.

Under $(\pi,\tilde{\pi}) \to (t\pi, t^{-1}\tilde{\pi})$, the $\psi = 0$ components of V_ϕ, V_f, and V_g scale as t^{-2}, t^{-4}, and 1, reflecting the fact that those operators describe states in Minkowski spacetime of helicities $1, 2$, and 0. In general, a state of helicity h is represented by a vertex operator that scales as t^{-2h}. The scattering amplitudes obtained by evaluating the expectation value of the product of vertex operators will likewise scale as t^{-2h}.

In a multi-particle scattering amplitude, let Φ be the set of external particles with vertex operators of type V_ϕ, F the set of external particles with vertex operators V_f, and G the set of external particles with vertex operators V_g. Let N_Φ, N_F, and N_G be the number of elements of Φ, F, and G, and let $N = N_\Phi + N_F + N_G$ be the total number of external particles. Let $p_k^{a\dot{a}} = \pi_k^a\tilde{\pi}_k^{\dot{a}}$ be the momentum of the k^{th} particle, let z_k be worldsheet coordinate at which it is asserted, and let $(\lambda_k,\mu_k,\psi_k) = (\lambda(z_k),\mu(z_k),\psi(z_k))$ parametrize the image of z_k in twistor space. Finally, let $u_k((\pi_k/\lambda_k)\psi_k)$ be the fermionic wavefunction of the k^{th} particle. In these definitions, k runs over all N possible values.

Now we commence to evaluate the scattering amplitudes. To evaluate the d^4x integral in (5.13), we use the fact that on the curve $\mathbb{D}_{x,\theta}$, $\mu^{\dot{a}} = x^{a\dot{a}}\lambda_a$. μ enters our wavefunctions only via the exponential factors, and therefore in any product of the above-described vertex operators of particles of momenta $p_j^{a\dot{a}} = \pi_j^a\tilde{\pi}_j^{\dot{a}}$, the x-dependence is of the form

$$\exp\left(ix_{a\dot{a}}\sum_j\pi_j^a\tilde{\pi}_j^{\dot{a}}\right). \tag{5.21}$$

We have used the delta functions in the wavefunctions to set λ^a of the j^{th} particle to a multiple of π_j; the multiple conveniently cancels out because of the (π/λ) factor in the exponent in (5.18). This is a typical example of how those factors often disappear in calculations, by helping turn λ's into π's. The x integral in (5.13) therefore gives simply a delta function of energy-momentum conservation, $(2\pi)^4\delta^4(\sum_j p_j^{a\dot{a}})$.

We use the scaling symmetry of the homogeneous coordinates (λ, μ, ψ) of $\mathbb{CP}^{3|4}$ to set $\lambda^1 = 1$. With the open string worldsheet understood as the upper half z-plane, a degree one instanton has $\lambda^2 = (az+b)/(cz+d)$ for some real a, b, c, and d; as discussed in obtaining (5.13), we fix the SL$(2, \mathbb{R})$ invariance so that $\lambda^2 = z$. It follows, for example, that the factor $\lambda_a \partial \lambda^a$ in the vertex operators of type V_g is equal to 1. Indeed, ∂ is just $\partial/\partial z$, so with $(\lambda^1, \lambda^2) = (1, z)$, we get $(\partial\lambda^1, \partial\lambda^2) = (0, 1)$. Moreover,

$$\frac{\lambda_k}{\pi_k} = \frac{1}{\pi_k^1}, \qquad z_j - z_k = -\frac{\langle \pi_j, \pi_k \rangle}{\pi_j^1 \pi_k^1}. \tag{5.22}$$

The vertex operator for the j^{th} external particle contains a delta function $\delta(\langle \lambda, \pi_j \rangle)$, which now becomes $\delta(\pi_j^2 - z_j \pi_j^1)$. The z_j integrals can be done with the help of these delta functions, with with the result that the j^{th} particle is inserted at $z_j = \pi_j^2/\pi_j^1$, and the amplitude acquires a factor

$$\int dz_j \delta(\pi_j^2 - z_j \pi_j^1) = 1/\pi_j^1 = \left(\frac{\lambda_j}{\pi_j}\right). \tag{5.23}$$

Let π_Φ and z_Φ denote the collection of variables π_i and z_i for $i \in \Phi$. In evaluating the scattering amplitude, we must evaluate the current correlation function $\mathcal{J}_0(z_\Phi) = \langle \prod_{k \in \Phi} J^{r_k}(z_k) \rangle$. Here by translation invariance, $\mathcal{J}_0(z_\Phi)$ is a function only of the differences $z_i - z_j$, which are written in terms of π_k and λ_k in (5.22), and of the Lie algebra indices r_k. SL$(2, \mathbb{R})$ invariance implies, since the currents J^r have dimension 1, that $\mathcal{J}_0(z_\Phi)$ can be written

$$\mathcal{J}_0(z_\Phi) = \mathcal{J}(\pi_\Phi) \prod_{i \in \Phi} \left(\frac{\pi_i}{\lambda_i}\right)^2 \tag{5.24}$$

(times a group theory factor which we suppress) for some function $\mathcal{J}(\pi_\Phi)$ that is homogeneous of degree -2 in each π_i. For example, the most familiar case is the case that the gauge group G is a unitary group, and after arranging the particles in Φ in a definite cyclic order, say $1, 2, \ldots, N_\Phi$, we extract a single-trace amplitude. In this case,

$$\mathcal{J}(\pi_\Phi) = \prod_{i \in \Phi} \frac{1}{\langle i, i+1 \rangle}. \tag{5.25}$$

This is a familiar factor in MHV scattering amplitudes for gluons, first interpreted as a current correlation function by Nair [7].

For every $j \in F$, the corresponding vertex operator contains a factor $(\lambda_j/\pi_j)^3 \tilde{\pi}_j^{\dot{a}} Y_{\dot{a}}$. The factor of $(\lambda_j/\pi_j)^3$ is included here purely for convenience. After extracting it, the rest of the vertex operator V_f for this particle is proportional to (π_j/λ_j); this factor cancels the factor of (λ_j/π_j) that comes from the z_j integral in (5.23). For vertex operators V_g, a similar cancellation occurs more directly; the vertex operator for $j \in G$ is proportional to (π_j/λ_j), which cancels the factor coming from the z_j integral. Vertex operators V_ϕ are proportional instead to (λ_j/π_j), but in this case the current correlation function gives a factor of $(\pi_j/\lambda_j)^2$, as in (5.24); these factors combine to (π_j/λ_j), which again cancels the factor coming coming from the z_j integral.

Returning to the factor $(\lambda_j/\pi_j)^3 \tilde{\pi}_j^{\dot{a}} Y_{\dot{a}}$ in a vertex operator of type V_f, it must be evaluated using the contraction $\langle Y_{\dot{a}}(z)\mu^{\dot{b}}(z')\rangle = \delta_{\dot{a}}^{\dot{b}}/(z-z')$. The only μ's in the wavefunctions are in the exponentials, and upon evaluating the contraction, we get a factor

$$i\left(\frac{\lambda_j}{\pi_j}\right)^3 \sum_{k\neq j} \frac{[j,k](\pi_k/\lambda_k)}{z_j - z_k}\,. \tag{5.26}$$

The sum over k includes particles of all types (from Yang-Mills or gravity multiplets) with $k \neq j$. With the aid of (5.22), we rewrite (5.26) in the form

$$-i\sum_{k\neq j} \frac{[j,k]}{\langle j,k\rangle} \frac{(\pi_k^1)^2}{(\pi_j^1)^2}\,. \tag{5.27}$$

If we introduce a spinor ζ^a with components $\zeta^a = (0,1)$, we can write this factor as

$$-i\sum_{k\neq j} \frac{[j,k]\langle k,\zeta\rangle^2}{\langle j,k\rangle\langle j,\zeta\rangle^2}\,. \tag{5.28}$$

This formula has the amazing property of being independent of ζ as long as $\zeta \neq \pi_j$ (where it is ill-defined because the denominator vanishes), and therefore it is covariant though the intermediate steps in the derivation were not manifestly covariant. To see the ζ-independence, note that any change in ζ takes the form $\zeta \to v\zeta + w\pi_j$ for some scalars v and w. As (5.28) is homogeneous in ζ of degree zero, we can set $v = 1$. Under $\zeta \to \zeta + w\pi_j$, the denominator of (5.28) is invariant, as $\langle j,j\rangle = 0$. The numerator is also invariant, after summing over k, because by momentum conservation

$$\sum_{k\neq j}[j,k]\langle k,\zeta\rangle = \sum_{k\neq j}[j,k]\langle k,j\rangle = 0\,. \tag{5.29}$$

(Momentum conservation states that $\sum_k |k]\langle k| = 0$; in the sums in (5.29), the restriction to $k \neq j$ is immaterial since $[j,j] = 0$.) We can reduce (5.28) to a covariant formula by setting ζ to equal one of the π_k, and having done so, we can restore bose symmetry by averaging over choices of ζ. But it seems more illuminating to leave the expression in the form given here.

When we combine all this, we learn that the tree level MHV scattering amplitude for these fields is

$$(-i)^F \mathcal{J}(\pi_\Phi) \prod_{j\in F}\sum_{k\neq j} \frac{[j,k]\langle k,\zeta\rangle^2}{\langle j,k\rangle\langle j,\zeta\rangle^2} \int d^8\theta^{Aa} \prod_{m=1}^N u_m\left(\left(\frac{\pi_m}{\lambda_m}\right)\psi_m\right). \tag{5.30}$$

We recall that u_m is the fermionic part of the wavefunction of the m^{th} particle. This wavefunction is to be evaluated on the instanton configuration, that is, for $\psi_m^A = \theta^{Aa}\lambda_{m\,a}$.

The integral over the fermionic parameters θ^{Aa} of the instanton that appears here is familiar from evaluating MHV tree amplitudes of Yang-Mills theory. (That is the special case that all vertex operators are of type V_ϕ; the fermionic integral in (5.30) does not distinguish the different types of vertex operator.) Consider the illuminating special case that $u_m = 1$ for all values of m except two, while $u_m = \frac{1}{4!}\epsilon_{ABCD}\psi_m^A\psi_m^B\psi_m^C\psi_m^D(\pi_m/\lambda_m)^4$ for the remaining two cases, say $m = r$ and s. This means that all particles other than particles r and s have the maximum helicity in their multiplets, while particles r, s have the minimum helicity in their multiplets. The maximum helicity is 1, 2, or 0 for particles described by vertex operators V_ϕ, V_f, or V_g, and the minimum helicities are -1, 0, or -2.

In this type of example, the integral over the θ^{Aa} gives a factor of $\langle r, s\rangle^4$, which is a familiar factor in the MHV tree amplitudes of Yang-Mills theory. So at last, the scattering amplitude becomes

$$(-i)^F \mathcal{J}(\pi_\Phi)\langle r, s\rangle^4 \prod_{j\in F}\sum_{k\neq j}\frac{[j,k]\langle k,\zeta\rangle^2}{\langle j,k\rangle\langle j,\zeta\rangle^2}\,. \tag{5.31}$$

This formula describes the MHV tree amplitude for scattering of $N-2$ particles which are either gauge bosons of helicity 1, gravitons of helicity 2, or dilatonic scalars C, and two particles, labeled r and s, which are either gauge bosons of helicity -1, scalars $\overline{C}$, or gravitons of helicity -2.

While MHV tree level amplitudes for gauge boson scattering are functions only of π and not $\tilde{\pi}$, we see from (5.31) that MHV tree level amplitudes with supergravitons have a non-trivial but polynomial dependence on $\tilde{\pi}$. In the terminology of [3], this corresponds to scattering amplitudes with "derivative of a delta function" support on curves of degree one.

5.3 Comparison to B-model of $\mathbb{CP}^{3|4}$

Finally, we will briefly attempt to interpret the results of section 5.1 on cubic tree level couplings of supergravitons in terms of the alternative approach to twistor-string theory via the B-model of $\mathbb{CP}^{3|4}$.

We recall from section 2.2 that vertex operators $V_f = f^I Y_I$ correspond in complex twistor space to disturbances in the almost complex structure J. On the other hand, vertex operators $V_g = g_I\partial Z^I$ correspond to disturbances in the B-field b. In section 2.2, we described a tree level coupling $\int b\wedge N(J)\,\Omega$, where $N(J)$ is the Nijenhuis tensor. As $N(J)$ is a nonlinear function of J, this includes a bJ^2 coupling. That bJ^2 coupling is the complex twistor space analog of the degree zero $f^2 g$ coupling that we found in section 5.1.

In section 5.1, however, we also found a degree zero f^3 coupling. To what in complex twistor space does this correspond? It should correspond to a local coupling on $\mathbb{CP}^{3|4}$ (local because of the degree zero property) that contains a term nonlinear in J but independent of b.

A natural candidate for this term is the integral of a certain Chern-Simons $(0,3)$-form that we will describe presently. Once this form, which we will call $\omega_{CS}(J)$, is constructed, the interaction we want is

$$\int_{\mathbb{CP}^{3|4}}\omega_{CS}(J)\,\Omega\,. \tag{5.32}$$

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It is independent of b and nonlinear in J. When expanded around the standard $\mathbb{CP}^{3|4}$, it leads to the desired J^3 interaction.

To construct this Chern-Simons form, we first note that on any manifold, there is a bundle T^* of one-forms. On an almost complex manifold, we have a decomposition

$$T^* = T^*_{1,0} \oplus T^*_{0,1} \tag{5.33}$$

of T^* into the bundles of forms of type $(1,0)$ and $(0,1)$, respectively. The ∂ and $\overline{\partial}$ operators on zero-forms (or functions) are defined as follows: if f is a function, then define $\overline{\partial} f$ and ∂f by writing $df = \partial f \oplus \overline{\partial} f$, where ∂f and $\overline{\partial} f$ are of type $(1,0)$ and $(0,1)$, respectively. No integrability of J is required in any of these statements.

Similarly, the bundle of two-forms can be decomposed as the direct sum of bundles of forms of type $(2,0)$, $(1,1)$, and $(0,2)$. We let $\pi_{1,1}$ be the projection operator from all two-forms to forms of type $(1,1)$.

Now we want to define a $\overline{\partial}$ operator on $T^*_{1,0}$. This operator, which we will call $\overline{D}$, will map $(1,0)$-forms to $(1,1)$-forms. We define it to map a $(1,0)$-form λ to

$$\overline{D}\lambda = \pi_{1,1}(d\lambda)\,. \tag{5.34}$$

Suppose that f is a function. Then $d(f\lambda) = f\,d\lambda + df \wedge \lambda$. The above definition of $\overline{D}$ leads to

$$\overline{D}(f\lambda) = \overline{\partial} f \wedge \lambda + f\,\overline{D}\lambda\,. \tag{5.35}$$

This is the defining property of a connnection, or in this case, of the $(0,1)$ part of a connection. It means that locally

$$\overline{D} = d\overline{X}^{\overline{I}} \left(\frac{\partial}{\partial \overline{X}^{\overline{I}}} + \alpha_{\overline{I}} \right) \tag{5.36}$$

for some $\alpha_{\overline{I}}$. $\overline{D}$ was defined to act on $T^*_{1,0}$, so the components of $\alpha_{\overline{I}}$ are matrices acting on $T^*_{1,0}$ (that is, they are endomorphisms of $T^*_{1,0}$). Thus, $\alpha_{\overline{I}}$ is a "gauge field," or at least the $(0,1)$ part of one. So one can construct from $\alpha_{\overline{I}}$ a Chern-Simons $(0,3)$-form in the standard fashion:

$$\omega_{CS}(J) = d\overline{X}^{\overline{I}} d\overline{X}^{\overline{J}} d\overline{X}^{\overline{K}}\ \mathrm{Tr}_{T^*_{1,0}} \left(\alpha_{\overline{I}} \partial_{\overline{J}} \alpha_{\overline{K}} + \frac{2}{3} \alpha_{\overline{I}} \alpha_{\overline{J}} \alpha_{\overline{K}} \right). \tag{5.37}$$

This $(0,3)$-form is then used in (5.32) to construct the desired interaction.

6. Conformal supergravity action

As we explained at the end of section 2, the contribution of an instanton of degree d to a scattering amplitude is proportional to $\exp(-d\langle C\rangle)$, where C is the lowest component of the superfield $\mathcal{W}$, and $\langle C\rangle$ is its expectation value.

Consider now an L-loop twistor-string amplitude for the scattering of N external gluons and gravitons. (Even when there are no external gravitons, this amplitude will not

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necessarily coincide with a similar amplitude in supersymmetric Yang-Mills theory, because supergravitons will appear as intermediate states.) According to [3], degree d curves of genus L contribute to N-gluon scattering processes with precisely $d+1-L$ gluons of negative helicity, and therefore $N-1-d+L$ positive helicity gluons.[9] Like the positive and negative helicity gluon, the positive and negative helicity graviton appears in the twistor fields at order ψ^0 and ψ^4. The selection rules for scattering amplitudes with positive and negative helicity particles are therefore the same whether the particles are gluons or gravitons. So L-loop amplitudes involving N_- negative-helicity gluons and gravitons depend on $\langle C\rangle$ as $\exp(-(N_-+L-1)\langle C\rangle)$. By parity symmetry, this implies that amplitudes involving N_+ positive-helicity gluons and gravitons depend on $\langle\overline{C}\rangle$ as $\exp\left(-(N_+ + L-1)\langle\overline{C}\rangle\right)$. (Of course, this result, like parity symmetry itself, is much less obvious in the twistor formalism.) Combining these results implies that L-loop scattering amplitudes for N gluons and gravitons depend on $\langle C\rangle$ and $\langle\overline{C}\rangle$ as

$$\exp\left(-(2L+N-2)\left\langle\frac{C+\overline{C}}{2}\right\rangle + h\left\langle\frac{C-\overline{C}}{2}\right\rangle\right) \tag{6.1}$$

where $h = N_+ - N_-$.

The L-loop amplitude for scattering of N gluons and gravitons further depends on the string coupling constant g_s as $(g_s)^{2L+N-2}$. A look at (6.1) shows that g_s can be absorbed into the expectation value of C and $\overline{C}$ by shifting $C \to C + \log(g_s)$ and $\overline{C} \to \overline{C} + \log(g_s)$.

What kind of conformal supergravity action would be consistent with this behavior? As we have exploited in section 4.3, the linearized $\mathcal{N}=4$ conformal supergravity can be described in terms of a chiral superfield $\mathcal{W}$ which is posited to obey the constraint (4.1). To get to the nonlinear level, one should introduce some suitable potentials from which a suitable nonlinear version of $\mathcal{W}$ can be constructed, in such a way that a constraint generalizing (4.1) emerges as a Bianchi identity. To our knowledge, this has not been done. If it can be done, and a suitable chiral superspace measure $E(x,\theta)$ can be constructed from the underlying potentials (and is invariant under constant shifts in C), then a supergravity action with the property we want might take the form

$$\mathcal{S} = \int d^4x \int d^8\theta\, E(x,\theta) e^{2\mathcal{W}(x,\theta)} + \int d^4x \int d^8\overline{\theta}\, \overline{E}(\widehat{x},\overline{\theta}) e^{2\overline{\mathcal{W}}(\widehat{x},\overline{\theta})}\,. \tag{6.2}$$

If the linearized theory is a good guide, $\mathcal{W}$ has zero conformal weight, so assuming the existence of a chiral superspace measure, the action

$$\mathcal{S} = \int d^4x \int d^8\theta\, E(x,\theta) f(\mathcal{W}) + \int d^4x \int d^8\overline{\theta}\, \overline{E}(x,\overline{\theta}) \overline{f}(\overline{\mathcal{W}}) \tag{6.3}$$

is supersymmetric for any holomorphic function $f(\mathcal{W})$. This is analogous to the case of $\mathcal{N}=2$ Yang-Mills theory, where for any gauge-invariant function $f_{YM}(\mathcal{W}_{YM})$ of the

[9] We consider only connected instantons; it has become reasonably clear [18, 19, 20] that the full connected twistor-string amplitudes can be computed just from these contributions, a fact which is also manifest in the open-string approach [4] since in that approach there are no disconnected instantons. It has also become fairly clear [19, 20] that the same amplitudes can be computed from totally disconnected instantons.

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adjoint-valued chiral superfield $\mathcal{W}_{YM}$, the action

$$\int d^4x d^4\theta\, f_{YM}(\mathcal{W}_{YM}) + \int d^4x d^4\overline{\theta}\, \overline{f}_{YM}(\overline{\mathcal{W}}_{YM}) \tag{6.4}$$

is supersymmetric.

In the "minimal" version of $\mathcal{N}=4$ conformal supergravity, which was first proposed in [14], the classical action was assumed to be invariant under a global $\mathrm{SL}(2,\mathbb{R})$ symmetry, which includes invariance under constant shifts of C. This would imply that $f(\mathcal{W}) = \mathcal{W}^2$. This choice is indeed analogous to the choice most often made in $\mathcal{N}=2$ super Yang-Mills theory, where the minimal theory (which in fact is renormalizable) arises if f_{YM} is quadratic.[10]

By contrast, it appears that twistor-string theory corresponds to $f(\mathcal{W}) = e^{2\mathcal{W}}$. With this choice of f, we lose the $\mathrm{SL}(2,\mathbb{R})$ symmetry that the classical theory very plausibly possesses if f is quadratic. However, it is conceivable that twistor-string theory has an $\mathrm{SL}(2,\mathbb{Z})$ symmetry. Possibly, such a symmetry acts as $\mathrm{SL}(2,\mathbb{Z})$ on the pair (W, W_D), where $W_D = \partial f/\partial W$; this would be roughly analogous to what happens in four-dimensional gauge theories with $\mathcal{N}=2$ supersymmetry [23]. Such a duality symmetry would entail non-classical electric-magnetic duality transformations on the metric tensor in spacetime, and in that respect would differ from presently known dualities in gauge theory and string theory.

7. Anomalies and gauge groups

In this section, we consider constraints associated with anomalies. First we consider implications of anomalies for the gauge group, and then we analyze some issues involving anomalies that are special to the $\mathbb{CP}^{3|4}$ approach to twistor-string theory.

7.1 Constraints on the gauge group

For physical type-I and heterotic strings, the possible gauge groups are determined by Green-Schwarz anomaly cancellation, or alternatively, by cancellation of certain worldsheet tadpoles and/or anomalies. What happens in twistor-string theory?

Here, there are two obvious constraints on the gauge group. They appear to lead to somewhat different answers, a point that at the moment we cannot illuminate.

One constraint arises because of the $\mathrm{SU}(4)$ R-symmetry of $\mathcal{N}=4$ super Yang-Mills theory. This symmetry is gauged when the super Yang-Mills theory is coupled to conformal supergravity; indeed, the $\mathrm{SU}(4)_R$ gauge fields, with helicities ± 1 and transforming in the **15** of $\mathrm{SU}(4)_R$, are part of the spectrum that we analyzed in sections 2 and 3.

[10] As briefly discussed in footnote *(c)* of [21], one way to construct a minimal action for $\mathcal{N}=4$ conformal supergravity is as follows. Couple $\mathcal{N}=4$ super Yang-Mills theory to background fields of conformal supergravity; compute the one-loop effective action for these background fields and extract its Weyl anomaly (that is, the change in the effective action under a global rescaling of the background metric [22]). Though the one-loop effective action is non-local and is not Weyl-invariant, the conformal anomaly is local and Weyl-invariant. Since it is also supersymmetric, it has the full local superconformal symmetry. Moreover, given the structure of the one-loop conformal anomaly, the part of this functional that involves the Weyl tensor is simply $\int d^4x\sqrt{g}\,W^2$, rather than $\int d^4x\sqrt{g}\,h(C,\overline{C})W^2$ for some non-trivial function $h(C,\overline{C})$.

The $\mathrm{SU}(4)_R$ symmetry is potentially anomalous. For example, in the vector multiplet, the massless helicity 1/2 fields transform as $\overline{\mathbf{4}}$ while the helicity $-1/2$ fields transform as the $\mathbf{4}$. So the vector multiplets give an $\mathrm{SU}(4)_R^3$ anomaly with coefficient $-\dim G$, with $\dim G$ the dimension of the gauge group G. By contrast, as analyzed in [24], the conformal supergravity multiplet has anomaly $+4$.[11] The authors of [24] threfore conclude that an anomaly-free theory must have $\dim G = 4$, so $G = \mathrm{SU}(2) \times \mathrm{U}(1)$ or $\mathrm{U}(1)^4$.

On the other hand, we can approach the matter using the worldsheet conformal anomaly. In the open-string approach to twistor-string theory, gauge symmetry arises when the matter system (whose lagrangian is called S_C in (2.1)) includes a current algebra. The only known general restriction on this matter system is that [4] it must have $c = 28$. This permits a wide variety of current algebras, and certainly does not determine what the dimension or rank of the gauge group must be. $\mathrm{SU}(2) \times \mathrm{U}(1)$ and $\mathrm{U}(1)^4$ are possible, but are not forced upon us.[12] Conceivably, from an open-string point of view, the $c = 28$ constraint must be supplemented by additional, presently unknown, restrictions involving tadpole cancellation in perturbation theory.

Apart from the $\mathrm{SU}(4)_R$ anomaly, one should consider other potential anomalies such as the conformal anomaly [2, 26]. We would presume, but have not demonstrated, that these anomalies are all related by $\mathcal{N} = 4$ supersymmetry, and vanish when the $\mathrm{SU}(4)_R^3$ anomaly does. If this is so, then as the $\mathrm{SU}(4)_R^3$ anomaly is a chiral anomaly that arises only in one-loop order, cancellation of this anomaly presumably entails cancellation of the conformal anomalies and other anomalies to all orders.

The level of the current algebra. Requiring $c = 28$ does not determine the symmetry group G of the current algebra, and likewise does not determine the *level* of the current algebra, which we will call k. For $\mathrm{SU}(2)$, a current algebra at level k has $c_{\mathrm{SU}(2)} = 3k/(k+2)$, so there is "room" for many values of k, even if we want the $c = 28$ system to be unitary.

In the discussion of scattering amplitudes in section 5, k really only enters because the connected part of the correlation function

$$\langle J^{r_1}(z_1) \dots J^{r_s}(z_s) \rangle \tag{7.1}$$

is proportional to k. Because of this, just as for the heterotic string, the Yang-Mills effective action in spacetime is proportional to k. (The ten-dimensional heterotic string has $k = 1$, but compactified models can be constructed with various values of k.) The kinetic energy for gauge and gravitational fields is thus qualitatively

$$\frac{1}{g_s^2} \int d^4x \left(k \,\mathrm{Tr}\, F^2 + W^2 \right), \tag{7.2}$$

[11] Our convention for SU(4) quantum numbers of fields and hence for the sign of the anomaly is opposite to that in [24].

[12] One exotic possibility is that the current algebra is described by the product of $\mathbb{T}^4$ and the $c = 24$ "Monstrous moonshine" conformal field theory of [25]. Since the monstrous moonshine conformal field theory has no dimension one fields, the only contribution to the spectrum would come from the four free bosons of $\mathbb{T}^4$ which could be used to construct vertex operators for $\mathrm{U}(1)^4$ gauge fields.

with g_s the string coupling constant, F the Yang-Mills field strength, and W the Weyl curvature. As explained in section 5, g_s is best understood as arising from the expectation value of the dilaton field, but this is not important at the moment. From (7.2), we see that to decouple conformal supergravity, we should take the limit $k \to 0$ with k/g_s^2 fixed.

Another way to explain why conformal supergravity decouples for $k \to 0$ is to note the following. We express the argument for the case $G = \mathrm{U}(N)$. In $\mathrm{U}(N)$ current algebra in genus zero, the single-trace part of the current correlators (7.1) are proportional to k, while multi-trace contributions are proportional to higher powers of k. So for $k \to 0$, if we adjust g_s^2 to cancel one power of k, the correlation functions reduce to single-trace expressions. The single-trace part of the genus zero correlation functions reproduce Yang-Mills scattering amplitudes, while (as explained in section 5.1 of [3]) multi-trace contributions to correlation functions reflect contributions from exchange of supergravitons.

Unitarity of the current algebra requires k to be a positive integer, and even if one does not care about unitarity, for the current algebra to be defined globally normally requires that k should be an integer. So at the moment it is difficult to see how to make sense in the string theory of the limit $k \to 0$, k/g_s^2 fixed.

7.2 Anomalies in the B-model of $\mathbb{CP}^{3|4}$

Now we briefly consider some issues involving anomalies in the other approach to twistor-string theory.

The first question we ask is this: what is the analog for the B-model of $\mathbb{CP}^{3|4}$ of the constraint $c = 28$?[13]

In this B-model, scattering amplitudes are computed by integration over a suitable moduli space $\mathcal{M}$ of curves $\mathbb{D} \subset \mathbb{CP}^{3|4}$. $\mathcal{M}$ is a complex manifold. To integrate over $\mathcal{M}$, one needs to find a suitable holomorphic measure Θ on $\mathcal{M}$. Then the integration is defined as $\int_{\mathcal{M}_{\mathbb{R}}} \Theta$, where $\mathcal{M}_{\mathbb{R}}$ is a suitable real cycle in $\mathcal{M}$.

We consider the case of N $D5$-branes on $\mathbb{CP}^{3|4}$, corresponding to gauge group $\mathrm{U}(N)$. We have to integrate over the space of triples consisting of

(i) a Riemann surface $\mathbb{D}$;

(ii) a holomorphic map $\Phi : \mathbb{D} \to \mathbb{CP}^{3|4}$;

(iii) and worldsheet fermions α^i, β_j, $i, j = 1, \ldots, N$ transforming under $\mathrm{U}(N)$ as $\mathbf{N}$ and $\overline{\mathbf{N}}$.

Suppose that we were merely trying to integrate over the moduli space $\mathcal{M}_0$ of abstract Riemann surfaces $\mathbb{D}$. There is no natural holomorphic measure, roughly speaking since $\mathcal{M}_0$ is not a Calabi-Yau manifold. A holomorphic measure would be a holomorphic section of the canonical bundle $K_{\mathcal{M}_0}$ of $\mathcal{M}_0$, but this line bundle is non-trivial. Instead, holomorphic factorization of the bosonic string [27] is based on the isomorphism $K_{\mathcal{M}_0} \cong \lambda_0^{13}$, where λ_0

[13] These issues have also been investigated by M. Movshev (private communication) with similar conclusions.

is the determinant line bundle of the $\overline{\partial}$ operator acting on ordinary functions. Existence of this isomorphism is equivalent to the statement that

$$K_{\mathcal{M}_0} \otimes \lambda_0^{-13} \tag{7.3}$$

is trivial and so has a natural holomorphic section. The measure of the $b-c$ ghost system of the bosonic string is a section of $K_{\mathcal{M}_0}$. The chiral part of the measure of a complex boson (or two real bosons) is a section of λ_0^{-13}. So by including 13 complex bosons, or more generally a chiral matter system with $c = 26$, we do obtain a holomorphic measure.

If the integration data were purely the Riemann surface $\mathbb{D}$ and the $\alpha - \beta$ system, that is (i) and (iii) above, then we would infer from this that the $\alpha - \beta$ system should have $c = 26$. Actually, as we explain momentarily, part (ii) of the data, the holomorphic map $\Phi : \mathbb{D} \to \mathbb{CP}^{3|4}$, effectively carries $c = -2$, so the $\alpha - \beta$ system should have $c = 28$. As a single pair of fermions α, β has $c = 1$, this seems to mean that we should take $N = 28$, leading to U(28) gauge theory at level one coupled to conformal supergravity. Of course, from the standpoint of the open-string approach to twistor-string theory, U(28) at level one is just one of many possibilities.

Given an abstract Riemann surface $\mathbb{D}$, we will now examine the holomorphic maps $\Phi : \mathbb{D} \to \mathbb{CP}^{3|4}$ of sufficiently high degree d. (For maps of low degree, the conclusion we will reach is still valid, but a more detailed argument is needed.) The key is to introduce the line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^{3|4}$, and define a line bundle $\mathcal{L}$ over $\mathbb{D}$ as the pullback $\mathcal{L} = \Phi^*(\mathcal{O}(1))$. For sufficiently high d, $\mathcal{L}$ can be any line bundle over $\mathbb{D}$ of degree d. Once $\mathcal{L}$ is picked, the map Φ is determined by picking four bosonic holomorphic sections $(s^a, s^{\dot{a}})$ of $\mathcal{L}$ and four fermionic ones s^A; Φ is then defined by setting $Z^I = s^I$. The possible Φ's for given $\mathcal{L}$ are parametrized by the choices of s^I modulo an overall scaling $s^I \to ts^I$, $t \in \mathbb{C}^*$. The space of s^I modulo this scaling is a copy of $\mathbb{CP}^{M-1|M}$ (for some M) and has a natural holomorphic measure, simply because the bosonic and fermionic variables in s^I are equal in number and have the same quantum numbers. (This statement is explained in more detail in section 4.6 of [3], where it is used to construct the integration measure for the case that $\mathbb{D}$ has genus zero.) So choosing a measure for integrating over maps $\Phi : \mathbb{D} \to \mathbb{CP}^{3|4}$ for fixed $\mathbb{D}$ amounts to choosing a measure for the space of line bundles $\mathcal{L}$.

The tangent space to the space of $\mathcal{L}$'s is $H^1(\mathbb{D}, \mathcal{O})$, so a measure on the space of $\mathcal{L}$'s is a section of the line bundle (over $\mathcal{M}_0$) whose fiber is $H^1(\mathbb{D}, \mathcal{O})^{-1}$. This is the bundle which above was called λ_0. Since the measure on the variables of type (ii) is thus effectively a holomorphic section of λ_0, to use the triviality of $K_{\mathcal{M}_0} \otimes \lambda_0^{-13}$ in constructing a measure for the overall system, the integration measure for the "matter" system must be a section of λ_0^{-14}, that is, the matter system must have $c = 28$.

The choice here of $\mathcal{L}$ is related in the open string analysis to the $GL(1)$ scaling, reviewed in section 2, which similarly (according to [4]) shifts the central charge required for the matter system from $c = 26$ to $c = 28$. The $D1$-brane in $\mathbb{CP}^{3|4}$ also carries a U(1) gauge field (from the $D1 - D1$ strings) that has no obvious counterpart in the open-string approach to twistor-strings, and which we have neglected above. Its role really merits further study.

A holomorphic Green-Schwarz mechanism. For the heterotic string, at least in the context of compactification, there are really two stages to worldsheet anomaly cancellation. Via the Green-Schwarz mechanism, worldsheet gauge and gravitational anomalies are canceled. Once this is done, it makes sense to discuss the c-number conformal anomaly.

Both of these issues have analogs for the B-model of $\mathbb{CP}^{3|4}$. We have already explored the analog of the conformal anomaly. Now let us briefly discuss the analogs of worldsheet gauge and gravitational anomalies.

The fermions α and β on the D-instanton worldvolume $\mathbb{D}$ have an action $\int_{\mathbb{D}}\beta\overline{\partial}_A\alpha$, where $\overline{\partial}_A=\overline{\partial}+A$ is the chiral Dirac operator acting on α. Here A is the $\mathrm{U}(N)$ gauge field on $\mathbb{CP}^{3|4}$, or rather its restriction or pullback to $\mathbb{D}$. Accordingly, the path integral over α and β gives a factor $\det\overline{\partial}_A$. Since $\overline{\partial}_A$ is a chiral Dirac operator, this determinant has a gauge anomaly. Under a gauge transformation $\delta A=\overline{\partial}_A\epsilon$, with ϵ an infinitesimal gauge parameter, we have

$$\det\overline{\partial}_A\to\exp\left(\frac{1}{2\pi}\int_{\mathbb{D}}d\overline{z}\wedge dz\ \mathrm{Tr}\ A_{\overline{z}}\partial_z\epsilon\right)\det\overline{\partial}_A\,. \tag{7.4}$$

Here z is a local holomorphic coordinate on $\mathbb{D}$. So gauge invariance appears to be lost.

What saves the day, just as for the heterotic string, is the coupling to the B-field. The worldsheet path integral is more accurately represented as

$$\exp\left(-\int_{\mathbb{D}}b_{z\overline{z}}dz\wedge d\overline{z}\right)\det\overline{\partial}_A\,, \tag{7.5}$$

so all is well if under gauge transformations

$$b_{I\overline{J}}\to b_{I\overline{J}}+\frac{1}{2\pi}\mathrm{Tr}\ A_{\overline{J}}\partial_I\epsilon\,. \tag{7.6}$$

This gauge transformation law for b can be deduced on other grounds. Let us consider the Chern-Simons $(0,3)$-form action for A [28, 3],[14]

$$\int d\overline{X}^{\overline{I}}\wedge d\overline{X}^{\overline{J}}\wedge d\overline{X}^{\overline{K}}\ \mathrm{Tr}\left(A_{\overline{I}}\partial_{\overline{J}}A_{\overline{K}}+\frac{2}{3}A_{\overline{I}}A_{\overline{J}}A_{\overline{K}}\right)\Omega\,. \tag{7.7}$$

(Local coordinates X^I are used here as in (2.14).) This action can be defined for any almost complex structure J. In verifying that it is gauge invariant, one uses $\overline{\partial}^2=0$, which only holds for integrable complex structures. In general, if the Nijenhuis tensor $N(J)$ is nonzero, the gauge variation of (7.7) is

$$\int d\overline{X}^{\overline{I}}d\overline{X}^{\overline{J}}d\overline{X}^{\overline{K}}\ \mathrm{Tr}\ \left(A_{\overline{I}}\partial_L\epsilon\right)N^L_{\overline{J}\,\overline{K}}\,\Omega\,. \tag{7.8}$$

In our discussion of the closed string modes in section 2.2, we did not take $N(J)=0$ by definition. Rather, $N(J)=0$ is the equation of motion for the field b, derived from the action (2.14). If we postulate that b transforms under gauge transformations as in (7.6)

[14] We will not be precise with our coefficients, so from this point of view, we will not verify the coefficient in (7.6).

(and adjust a couple of coefficients), the sum of (2.14) and (7.7) becomes gauge-invariant. Thus, the gauge transformation law of b that restores gauge-invariance in the D-instanton path integral is also needed to ensure gauge-invariance of the bulk effective action on $\mathbb{CP}^{3|4}$.

There is apparently a similar story for diffeomorphism anomalies. The Chern-Simons action $\int \omega_{CS}(J)\Omega$ of eq. (5.32) is diffeomorphism invariant only if $N(J) = 0$. But the sum of (2.14) and (5.32) is diffeomorphism-invariant if one adds a gravitational contribution to the transformation law (7.6) of b. This contribution is analogous to the transformation of the B-field of the heterotic string under diffeomorphisms, and also serves to cancel the gravitational anomaly in the D-instanton measure.

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